BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY

A Historical and Critical Review of Mathematical Science

EDITED BY

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VOL. XX

October 1913 to July 1914

PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK
1914

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OCTOBER, 1913

Whole No. 221

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BULLETIN OF THE

AMERICAN MATHEMATICAL SOCIETY.

NOTE ON THE GAMMA FUNCTION.

BY PROFESSOR GEORGE D. BIRKHOFF.

(Read before the American Mathematical Society, April 26, 1913.)

IN this note I wish to present a simple development of the principal properties of the function $\Gamma(x)$, based on the elements of the theory of functions of a complex variable.

1. Let $\varphi(x)$ denote the function

$$x^{x-\frac{1}{2}}e^{-x}\sqrt{2\pi} \quad (-\pi < \arg x < +\pi), \tag{1}$$

where that determination of $\varphi(x)$ is chosen which is real and positive when the complex variable x is real and positive. The function $\Gamma(x)$ is defined to be

$$\lim_{n=\infty} \frac{1}{x}\cdot\frac{1}{x+1}\cdots\frac{1}{x+n}\varphi(x+n+1) \tag{2}$$

$$(n = 0, 1, 2, \cdots;\ x \neq 0, -1, -2, \cdots).*$$

It is necessary to establish first that the limit of the sequence exists. Denote the $(n+1)$th term of the sequence by $p_n(x)$. It is clear that none of these terms vanish, and that the question of convergence of the sequence is essentially the same as of the series

$$\log p_0(x) + \log [p_1(x)/p_0(x)] + \log [p_2(x)/p_1(x)] + \cdots, \tag{3}$$

where the principal logarithms are taken.

We have at once the relation

$$\frac{p_n(x)}{p_{n-1}(x)} = \frac{\varphi(x+n+1)}{(x+n)\varphi(x+n)}. \tag{4}$$

* A similar formula has been obtained by Enneper, Dissertation, Göttingen (1856), p. 10.

Let us therefore consider the function $\log[\varphi(x+1)/x\varphi(x)]$, since from it may be obtained the $(n+1)$th term of (3) by replacing x by $x+n$. A substitution of the known value of $\varphi(x)$ gives us at once

$$\log\frac{\varphi(x+1)}{x\varphi(x)} = \left(x+\frac{1}{2}\right)\log\left(1+\frac{1}{x}\right) - 1$$
$$= \left(\frac{1}{3}-\frac{1}{2\cdot 2}\right)\frac{1}{x^2} - \left(\frac{1}{4}-\frac{1}{2\cdot 3}\right)\frac{1}{x^3} + \cdots,$$

the last member being the expansion of the function in powers of $1/x$ in the vicinity of $x=\infty$. This expansion converges for $|x|>1$ since the function has no singularity save two branch points of infinite order at 0 and -1; the expansion in series will converge uniformly for $|x| \geqq d > 1$, and the function it represents may be written $M(x)/x^2$, where $M(x)$ remains finite and analytic for $|x| \geqq d$. That is, we have

$$\log\frac{\varphi(x+1)}{x\varphi(x)} = \frac{M(x)}{x^2} \quad (|M(x)| \leqq K \text{ if } |x| \geqq d).$$

The series (3) may now be written

$$(3') \quad \log\varphi(x) + \frac{M(x)}{x^2} + \frac{M(x+1)}{(x+1)^2} + \frac{M(x+2)}{(x+2)^2} + \cdots.$$

Suppose first that x lies in the right half-plane and also that $|x| \geqq d$. In this case $x+1$, $x+2$, $\cdots$ will exceed d in absolute value, and the $(n+1)$th term of (3′) will be not greater in absolute value than the nth term of the series of positive quantities

$$(5) \qquad \frac{K}{|x|^2} + \frac{K}{|x+1|^2} + \frac{K}{|x+2|^2} + \cdots.$$

For x in the right half-plane we have $u \geqq 0$, if $x = u + \sqrt{-1}v$, so that

$$|x+n|^2 = (u+n)^2 + v^2 \geqq u^2 + v^2 + n^2 = |x|^2 + n^2.$$

Accordingly each term of the above series is not greater in absolute value than the corresponding term of

$$(6) \qquad \frac{K}{|x|^2} + \frac{K}{|x|^2+1^2} + \frac{K}{|x|^2+2^2} + \cdots.$$

If we put $x = d$ in this series we do not decrease the absolute value of any term, and obtain a convergent series of positive constants. Therefore by Weierstrass's test the original series (3) converges absolutely and uniformly for $|x| \geqq d$ and x in the right half-plane. This demonstrates that the limit (2) exists, is analytic, and does not vanish, for x restricted in the manner stated.

By definition of $p_n(x)$ we obtain for $k = 0, 1, \cdots$,

$$p_n(x) = \frac{1}{x} \cdot \frac{1}{x+1} \cdots \frac{1}{x+k}\, p_{n-k-1}(x+k+1) \qquad (x \neq 0, -1, \cdots). \tag{7}$$

Also for any x we may choose k so large that $x + k + 1$ lies to the right of the imaginary axis, and exceeds d in absolute value. From what precedes we see that the last factor on the right in (7) will uniformly approach the limit $\Gamma(x+k+1)$ different from zero, in the vicinity of this x, as n becomes infinite. Thus the limit (2) exists in all cases and represents a function $\Gamma(x)$ nowhere zero, analytic in the entire plane with the exception of the points $0, -1, \cdots$. It is clear from the formula (7) that at these points $\Gamma(x)$ has a pole of the first order.

If in particular we put $k = 0$ in (7) and let n become infinite we see that $\Gamma(x)$ is a solution of the functional equation in $f(x)$

$$f(x+1) = xf(x). \tag{8}$$

The function $\Gamma(x)$ *given by* (2) *is, for* $x \neq 0, -1, -2, \cdots$, *a single-valued and analytic function different from zero and satisfying the relation* $\Gamma(x+1) = x\Gamma(x)$. *At the excluded points* $\Gamma(x)$ *has a pole of the first order.*

2. It remains to characterize $\Gamma(x)$ in the vicinity of $x = \infty$. For x in the right half-plane and $|x| \geqq d$, we have seen that the $(n+1)$th term of (3′) is not greater in absolute value than the nth term of (6); this in turn is not greater than the nth term of the series

$$\frac{K}{|x|^2} + K\int_0^1 \frac{dt}{|x|^2 + t^2} + K\int_1^2 \frac{dt}{|x|^2 + t^2} + \cdots,$$

whose sum evaluates to $K/|x|^2 + K\pi/2\,|x|$. In this way a simple upper limit of the form $K'/2\,|x|$ for the sum of the terms of (3′) after the first is obtained when $|x| \geqq d$.

For x in the left half-plane and for $|v| \geqq d$, all of the quantities $x+1, x+2, \cdots$, having the same imaginary component, are also at least as great as d in absolute value. Accordingly the $(n+1)$th term of (3′) is again not greater in absolute value than the nth term of (5). In this case the sum of the series (5) is less than $K'/|v|$. In fact, the terms after the leading nth one for which $x+n$ lies on or to the right of the imaginary axis constitute a series of the form before considered of sum less in absolute value than $K/|x+n|^2 + K\pi/2\,|x+n|$ and therefore less than $K'/2\,|v|$. The remaining terms constitute a part of a similar series

$$\frac{K}{|-x-n+1|^2} + \frac{K}{|-x-n+2|^2} + \cdots,$$

also less in absolute value than the same quantity.

If then l denotes the distance of the point x from the negative half of the real axis,* we find from (3′) that

$$\log \Gamma(x) = (x-\tfrac{1}{2}) \log x - x + \log \sqrt{2\pi} + \mu(x), \tag{9}$$
$$(|\mu(x)| < K'/l \text{ for } l \geqq d.)$$

This result gives at once the following: *the function* $\Gamma(x)$ *has the property that lim* $\Gamma(x)/\varphi(x) = 1$ *if the point* x *recedes indefinitely from the negative half of the real axis.*

3. The fact that $\Gamma(x)$ satisfies the functional equation (8) is of central importance. By means of this fact the function $e^{\pi\sqrt{-1}x}/\Gamma(1-x)$ can also be seen to be a solution of (8). The ratio of $\Gamma(x)$ to this new solution is periodic of period 1, for if x is changed to $x+1$ the equation (8) shows that the first and second members of the ratio are both multiplied by x. This ratio

$$p(x) = \Gamma(x)\Gamma(1-x)/e^{\pi\sqrt{-1}x}$$

may also be verified directly to be of period 1.

Consider now $p(x)$ in the period strip $0 \leqq u \leqq 1$ where it has no singularities save a pole of the first order at $x = 0, 1$; it is clear from the definition of $p(x)$ that this periodic function nowhere vanishes in the strip.

* If $u \leqq 0$ we have $l = |v|$ and if $u \geqq 0$ we have $l = |x|$.

Let us determine the asymptotic form of $p(x)$ at the ends of the strip. We have by (9)

$$\log p(x) = (x - \tfrac{1}{2})[\log x - \log(1 - x)] - 1 + \log 2\pi + \mu(x) + \mu(1 - x) - \pi\sqrt{-1}x.$$

If x lies in the upper half of the strip, $1 - x$ lies in the lower half of the strip, and *vice versa*. Let x tend to infinity in the upper half of the strip, then $1 - x$ will tend to infinity in the lower half. By virtue of our convention we shall have

$$\log (1 - x) = -\pi\sqrt{-1} + \log(x - 1),$$

where $\log(x - 1)$ is the principal logarithm. Substituting, we find

$$\log p(x) = \left[-\left(x - \frac{1}{2}\right)\log\left(1 - \frac{1}{x}\right) - 1\right] - \frac{\pi\sqrt{-1}}{2} + \log 2\pi + \mu(x) + \mu(1 - x).$$

This equation proves that, as x becomes infinite in the upper end of the strip,

$$\lim p(x) = -2\pi\sqrt{-1},$$

since the first term and the last two terms on the right-hand side approach zero. Likewise, as x approaches infinity in the lower half-strip, we have

$$\log (1 - x) = \pi\sqrt{-1} + \log (x - 1)$$

and find

$$\lim e^{2\pi\sqrt{-1}x}p(x) = 2\pi\sqrt{-1}.$$

Accordingly if we write

$$z = e^{2\pi\sqrt{-1}x}, \quad p(x) = q(z),$$

the strip in the x-plane is transformed into the complete z-plane, the upper and lower ends of the strips corresponding to $z = 0$ and $z = \infty$ respectively; and at the same time it is evident that $q(z)$ is single-valued, analytic in the extended z-plane save at $z = 1(x = 0)$, where it has a pole of the first order; furthermore $q(z)$ vanishes nowhere save at $z = \infty$,

where $zq(z)$ takes the value $2\pi\sqrt{-1}$. These facts show at once that $q(z)$ is the rational function of z

$$\frac{2\pi\sqrt{-1}}{(z-1)}.$$

Substituting the corresponding value for $p(x)$ in the equation of definition, there results

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}, \tag{10}$$

a fundamental formula. If both sides are multiplied through by x, the left-hand side may be written $\Gamma(x+1)\Gamma(1-x)$ and tends to $\Gamma^2(1)$ as x tends to zero; the right-hand side tends to 1. Since, by (2), $\Gamma(1)$ is positive we obtain

$$\Gamma(1) = 1. \tag{11}$$

4. According to the results of section 2, we have the important relation

$$\lim_{l=\infty} \frac{\Gamma(x+y)}{x^y\Gamma(x)} = 1, \tag{12}$$

where y is fixed and l again represents the distance of the point x from the nearest point of the negative half of the axis of reals; in fact these results show that this limit is equal to

$$\lim_{l=\infty} \frac{\varphi(x+y)}{x^y\varphi(x)} = \lim_{l=\infty} \left(1+\frac{y}{x}\right)^{x+y-\frac{1}{2}} e^{-y} = 1. \tag{13}$$

Now if we divide $p_n(x)$ by $p_{n-1}(1)$ we find at once

$$\frac{p_n(x)}{p_{n-1}(1)} = \frac{1\cdot 2\cdots n}{x\cdot x+1\cdots x+n} \frac{\phi(x+n+1)}{\phi(n+1)};$$

if further we make use of equations (11) and (13), letting n become infinite, we obtain Euler's formula

$$\Gamma(x) = \lim_{n=\infty} \frac{1\cdot 2\cdots n}{x\cdot x+1\cdots x+n} (n+1)^x. \tag{14}$$

It is to be noted that the final factor on the right, which replaces $\varphi(x+n+1)/\varphi(n+1)$ in $p_n(x)/p_{n-1}(1)$, is not of such a nature as to affect the uniform convergence of the

logarithm of the sequence. This property is carried over from (3) since the final factor bears to the factor which it replaces a ratio which approaches 1 uniformly in any finite region as n becomes infinite.

If we introduce Euler's constant

$$C = \lim_{n=\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right),$$

we obtain the Schlömilch product formula

$$\Gamma(x) = \frac{e^{-Cx}}{x} \prod_1^\infty \frac{e^{x/n}}{1 + x/n}. \tag{15}$$

In this case the final factor on the right in (14) is replaced by

$$e^{\left(1+\frac{1}{2}+\cdots+\frac{1}{n}-c\right)x}.$$

5. Another form of solution of the functional equation is the definite integral

$$\int_0^{+\infty} e^{-t} t^{x-1} dt,$$

valid for x in the right half-plane. The fact that this integral $I(x)$ satisfies the functional equation may be verified by forming $xI(x)$ and integrating by parts. It is furthermore evident that $I(1) = 1$.

To prove that $I(x)$ is identical with $\Gamma(x)$ we can proceed as follows. The function $I(x)$ is analytic for any x within the right half-plane, since the integrand is analytic in x for $t > 0$; and the integral is absolutely and uniformly convergent in the vicinity by the ordinary tests. The ratio function

$$p(x) = I(x)/\Gamma(x)$$

is periodic of period 1 in x in consequence of the fact that $I(x)$ and $\Gamma(x)$ are solutions of (8); and this function $p(x)$ is analytic in x throughout any period strip such as $1 \leqq u \leqq 2$.

Now the path of integration along the positive half of the real axis may be modified to be any ray within the right half-plane from $t = 0$ to $t = \infty$. In fact the integrand is continuous in the sector formed by the positive half of the real axis and such a ray, and vanishes to infinite order at $t = \infty$; thus Cauchy's integral theorem may be applied to show that

the integral taken around the sector vanishes. Take the ray to pass through $t = x$ in the selected period strip and write $t = x\rho$; we obtain

$$I(x) = x^x \int_0^{+\infty_i} e^{-x\rho} \rho^{x-1} d\rho,$$

where ρ is a real variable. Hence we see that for $1 \leqq w \leqq 2$

$$|I(x)| \leqq |x^x| \left\{ \int_0^1 d\rho + \int_1^\infty e^{-\rho} \rho d\rho \right\} < 2|x^x|,$$

since if the second integral in the bracketed expression were taken from 0 to ∞ it would give $I(2) = I(1) = 1$ by (8) and (11). Moreover by (9) it is clear that

$$|\Gamma(x)| > |x^{x-1}| \quad \text{for} \quad 1 \leqq u \leqq 2, \quad |v| > \lambda,$$

if λ is sufficiently large and positive.

Consequently we obtain, for $|v|$ sufficiently large,

$$|p(x)| < 2|x|.$$

As before, write

$$z = e^{2\pi\sqrt{-1}x}, \quad p(x) = q(z).$$

It is apparent that $q^{(z)}$ is single-valued and analytic at every point save $z = 0$ and $z = \infty$, where however $zq(z)$ and $q(z)/z$ respectively tend to zero. It follows by Riemann's theorem that $q(z)$ is analytic in the extended plane, and thus is a constant which reduces to 1 since it has already been seen that $I(1) = \Gamma(1) = 1$.

6. Differentiating Schlömilch's infinite product logarithmically, we obtain the series for the Euler ψ-function

$$(16) \quad \psi(x) = \frac{d}{dx} \log \Gamma(x) = -C - \frac{1}{x} - \sum_1^\infty \left[\frac{1}{x+n} - \frac{1}{n} \right].$$

The differentiation is legitimate because of the uniformity of convergence of this product. From the properties of $\Gamma(x)$ it follows that $\psi(x)$ *is analytic in the finite plane save for poles of the first order at* $0, -1, \ldots$ *with residues* -1 *at these points.*

We furthermore obtain

$$(17) \quad \psi(x+1) - \psi(x) = \frac{1}{x}, \quad \psi(x) - \psi(1-x) = \pi \cot \pi x$$

directly from the functional equation for $\Gamma(x)$ and from (10).

To investigate the nature of $\psi(x)$ at $x = \infty$, we differentiate (9), which then becomes

$$\psi(x) = \log x - \frac{1}{2x} + \frac{d}{dx}\mu(x),$$

and consider the magnitude of the last term. Take a point x at a distance $l \geqq 2d$ from the negative half of the real axis, and about it draw a circle C of radius $l/2$. Then we have by differentiating Cauchy's integral for $\mu(x)$, and using the upper limit for $\mu(x)$ given in (9),

$$\left|\frac{d}{dx}\mu(x)\right| = \left|\frac{-1}{2\pi\sqrt{-1}}\int_C \frac{\mu(t)}{(t-x)^2}dt\right| \leqq \frac{2K'}{\pi l^2}\int_C |\,dt\,| = \frac{2K'}{l^2}.$$

This desired inequality shows that $d\mu/dx$ is of the second order in $1/l$ and that *lim $\psi(x)/\log x = 1$ as the distance of the point x from the negative half of the real axis becomes infinite in any manner.*

7. A final central theorem is the development of the function $\Gamma(x)$ relates to the evaluation of the beta function

$$B(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt, \tag{18}$$

where x and y have positive real parts. The formula to be proven is

$$B(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{19}$$

The attack is identical in spirit with that followed in section 5.

First we observe that $B(x, y)$ is analytic and symmetric in x, y in the domain under discussion and that $B(1, 1) = 1$. Also, integrating by parts, we find

$$xB(x, y+1) = yB(x+1, y),$$

and we find directly

$$\begin{aligned} B(x, y+1) &= \int_0^1 t^{x-1}(1-t)^{y-1}d - \int_0^1 t^x(1-t)^{y-1}dt \\ &= B(x, y) - B\,(x+1, y). \end{aligned}$$

Thus we obtain the functional equation in x

$$B(x+1, y) = \frac{xB(x, y)}{x+y},$$

and likewise the same functional equation in y. It is readily verified that the right-hand member of (19) satisfies these equations and hence that

$$p(x, y) = \frac{B(x, y)\Gamma(x+y)}{\Gamma(x)\Gamma(y)}$$

is a function periodic in x and y of period 1 and analytic in the domain under consideration.

Now hold y fixed of real part not less than 1, and let x be arbitrary in the period strip $1 \leqq u \leqq 2$. We have then

$$| B(x, y) | \leqq \int_0^1 | t^{x-1}(1-t)^{y-1} | \, dt \leqq 1.$$

Moreover we have for some $k > 0$

$$\left| \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \right| > k \left| \frac{\Gamma(x)}{\Gamma(x+y)} \right| > \frac{k}{2} | x^{-y} |$$

by (12) for $| v |$ large enough. Thus there results

$$| p(x, y) | < \frac{2|x^y|}{k}$$

for $| v |$ large enough. Since $p(x, y)$ is analytic throughout the finite strip in x, and since this relation shows that $p(x, y)$ is finite at both ends of the strip (see section 5), it follows that $p(x, y)$ is finite throughout the strip. Hence $p(x, y)$ is constant in x, and likewise in y, and therefore is constant in both variables, being equal to $p(1, 1) = 1$. This demonstrates the truth of (19).

The general type of consideration given above admits of further development, but the material which I have presented will serve to indicate its nature.

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SOME PROPERTIES OF SPACE CURVES MINIMIZING A DEFINITE INTEGRAL WITH DISCONTINUOUS INTEGRAND.

BY DR. E. J. MILES.

(Read before the American Mathematical Society, February 25, 1911.)

In a recent paper Bliss and Mason* considered the problem of the calculus of variations in which the integrand function is allowed to have a finite discontinuity along a given plane curve which separates the fixed end points. Later they made a systematic extension of the Weierstrassian theory of the calculus of variations to problems in space.† The object of the present note is to state the results obtained by applying the method used in the first mentioned paper to the case of a discontinuous integrand occurring in the space problem.

The problem studied may then be stated in the following way: Among all curves which go from the point 1 to the point 3 lying on opposite sides of a given surface S, and which cross S but once, it is required to find the one which minimizes the sum of the two integrals

$$J = \int F(x, y, z, x', y', z')dt, \quad j = \int f(x, y, z, x', y', z')dt,$$

the first integral to be taken from the point 1 to the surface S and the second from S to the point 3.

§ 1. *Equations Defining the Minimizing Curves.*

In order to find the equations of the minimizing curves

$$x = x(t), \quad y = y(t), \quad z = z(t), \tag{1}$$

it will be supposed that all curves considered lie in the interior of a region R of space. The surface S defined by the equations

$$S: \qquad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v)$$

* Bliss and Mason: "A problem of the calculus of variations in which the integrand is discontinuous," *Transactions*, vol. 7 (1906).

† Bliss and Mason: "The properties of curves in space which minimize a definite integral," *Transactions*, vol. 9 (1908). For brevity this paper will hereafter be referred to as I.

is supposed to have no singular points in R and divides this region into two regions R_1 and R_2 which contain the points 1 and 3 respectively. In R_1 the variables and their derivatives will be denoted by capital letters while in R_2 small letters will be used. Then in the whole of R it will be assumed that the two functions F and f have the properties ordinarily imposed in the space problem. Hence the functions (1) must satisfy the Euler differential equations and if the parameter is chosen as the length of arc and the problem assumed to be regular it follows that the extremals for J can therefore be written in the form

$$\begin{aligned} X &= \Phi(s; X_1, Y_1, Z_1, X_1', Y_1', Z_1'), \\ Y &= \Psi(s; X_1, Y_1, Z_1, X_1', Y_1', Z_1'), \\ Z &= \mathrm{X}(s; X_1\, Y_1, Z_1, X_1', Y_1', Z_1'), \end{aligned} \tag{2}$$

where

$$X_1'^2 + Y_1'^2 + Z_1'^2 = 1 \tag{3}$$

and the following initial conditions are satisfied

$$\begin{aligned} \Phi(0; X_1, Y_1, Z_1, X_1', Y_1', Z_1') &= X_1, \\ \Phi_s(0; X_1, Y_1, Z_1, X_1', Y_1', Z_1') &= X_1', \\ \Psi(0; X_1, Y_1, Z_1, X_1', Y_1', Z_1') &= Y_1, \\ \Psi_s(0; X_1, Y_1, Z_1, X_1', Y_1', Z_1') &= Y_1', \\ \mathrm{X}(0; X_1, Y_1, Z_1, X_1', Y_1', Z_1') &= Z_1, \\ \mathrm{X}_s(0; X_1, Y_1, Z_1, X_1', Y_1', Z_1') &= Z_1'. \end{aligned} \tag{4}$$

A similar set of extremals exists for the integral j which can be denoted by the equations

$$\begin{aligned} x &= \varphi(s; x_1, y_1, z_1, x_1', y_1', z_1'), \\ y &= \psi(s; x_1, y_1, z_1, x_1', y_1', z_1'), \\ z &= \chi(s; x_1, y_1, z_1, x_1', y_1', z_1'), \end{aligned} \tag{5}$$

where the functions φ, ψ, χ satisfy initial conditions similar to (4).

Let 2 be the point where the minimizing curve joining 1 and 3 intersects the surface S. Then since the arcs C_{12} and C_{23} must minimize the integrals J and j with respect to all curves joining 1 and 2, 2 and 3 respectively and lying in the regions R_1 and R_2 it follows that the regular conditions I and II are necessary:

I. *The arcs C_{12} and C_{23} are extremals for the integrals J and j, respectively, and therefore belong to the sets* (2) *and* (5).

II. *The functions F_1 and f_1 must be positive along the arcs C_{12} and C_{23} satisfying* I.

§ 2. *The Third Necessary Condition.*

The third necessary condition is found to be a restriction on the direction of the extremal arcs C_{12} and C_{23} at the point of intersection 2 with the surface S.

In order to find what this restriction is let it be supposed that the minimizing curve is imbedded in a one-parameter family of curves

$$x = a(t, u), \quad y = b(t, u) \quad z = c(t, u), \tag{6}$$

which has the following properties. The curves all intersect the surface S for $t = t_2$, pass through the point 1 for $t = t_1$, through 3 for $t = t_3$, and contain the curve C for $u = u_2$. Thus it has been assumed that the point 2 varies along the u-parameter curve of S. The sum of the two integrals J and j taken along any number of the set (6) from 1 to 3 is evidently a function of u which may be denoted by $I(u)$. But if C is to furnish a minimum the derivative $I'(u)$ must vanish when $u = u_2$. By the ordinary methods of variations this leads to the following result:

$$[F_{X'} - f_{x'}]x_u + [F_{Y'} - f_{y'}]y_u + [F_{Z'} - f_{z'}]z_u = 0, \tag{7}$$

where the arguments of the derivatives of F are the values of X, Y, Z, X', Y', Z' on the curve C_{12} at the point 2, while those of the derivatives of f are the values x, y, z, x', y', z' on C_{23} at 2 and x_u, y_u, z_u define the direction of the u-parameter line on S.

Likewise when the point 2 varies on the v-parameter line of the surface there results the corresponding equation

$$[F_{X'} - f_{x'}]x_v + [F_{Y'} - f_{y'}]y_v + [F_{Y'} - f_{z'}]z_v = 0. \tag{8}$$

The third condition is then

III. *The extremal arcs C_{12} and C_{23} together with the surface S must satisfy equations* (7) *and* (8) *at the intersection point* 2 *on the surface.*

Since the direction cosines of any curve on S are proportional to

$$x_u u' + x_v v', \quad y_u u' + y_v v', \quad z_u u' + z_v v',$$

it follows that equations (7) and (8) will be satisfied when the point moves on any curve passing through 2 and furthermore the bracketed quantities in these equations are proportional to the direction cosines of the normal to the surface at 2.

§ 3. *The System of Extremals for j Determined by Those of J and Condition III.*

It can now be shown that if a curve C_{123} has been found which satisfies conditions I, II and III then each extremal C_{12}, of the set (2) determines uniquely in connection with III one of the set (5).

In order to see this, think of equations (2) as a set of extremals through the fixed point (X_1, Y_1, Z_1) intersecting the surface at points $2'$. X_1', Y_1', Z_1' are then parameters satisfying the equation (3).

In place of the parameters X_1', Y_1', Z_1' it will be found advantageous to introduce the values of the u and v parameter lines of S. This can be accomplished by solving* the equations

$$\begin{aligned} &\Phi(s; X_1, Y_1, Z_1, X_1', Y_1', Z_1') = x(u, v), \\ &\Psi(s; X_1, Y_1, Z_1, X_1', Y_1', Z_1') = y(u, v), \\ &\mathrm{X}(s; X_1, Y_1, Z_1, X_1', Y_1', Z_1') = z(u, v), \\ &X_1'^2 + Y_1'^2 + Z_1'^2 = 1 \end{aligned} \tag{9}$$

for s, X_1', Y_1', Z_1' in terms of u and v and substituting the values of X_1', Y_1', Z_1' thus obtained in (2). Equations of the following form result

$$X = \Phi(s, u, v), \quad Y = \Psi(s, u, v), \quad Z = \mathrm{X}(s, u, v). \tag{10}$$

Consider now the equations

$$\begin{aligned} &x_2'^2 + y_2'^2 + z_2'^2 = 1, \\ &f_{x'}x_u + f_{y'}y_u + f_{z'}z_u = F_X x_u + F_Y y_u + F_Z z_u, \\ &f_{x'}x_v + f_{y'}y_v + f_{z'}z_v = F_X x_v + F_Y y_v + F_Z z_v, \end{aligned} \tag{11}$$

* This is always possible for points $2'$ on S sufficiently near 2, provided the point 2 on C_{12} is not conjugate to 1. See Bliss and Mason, I, p. 446 and following.

of which the last two are simply the equations (7) and (8) written in a separated form. By means of the equations of the surface S and (10) it is possible to express X_2, Y_2, Z_2; x_2, y_2, z_2; x_u, y_u, z_u; x_v, y_v, z_v and X_2', Y_2', Z_2' in terms of u and v. Equations (11) are then functions of x_2', y_2', z_2', u and v, having one solution corresponding to 2. Hence there will be a unique solution for x_2', y_2', z_2' as functions of u and v provided the functional determinant of the left hand members of (11) is different from zero.

This determinant is found to have the value*

$$c[x_2'\{f_x' - F_X'\} + y_2'\{f_y' - F_Y'\} + z_2'\{f_z' - F_Z'\}]$$
$$f_1(x_2'^2 + y_2'^2 + z_2'^2),$$

an expression which can vanish only when the term in brackets vanishes. This can happen only when the arc C_{23} is tangent to S at 2—a case which will be excluded. Hence there is a unique solution for x_2', y_2' and z_2' in terms of u and v, and when these values are substituted in equations (5) it is seen that they assume the form

$$(12) \quad x = \varphi(s, u, v), \quad y = \psi(s, u, v), \quad z = \chi(s, u, v),$$

where of course x_2, y_2, z_2 have also been replaced by the functions $x(u, v)$, $y(u, v)$, $z(u, v)$ defining the point 2 on S.

It has therefore been shown that *if C_{123} is a curve satisfying conditions* I, II, III, *then to each extremal through* 1, $C_{12'}$ *of the integral J and near C_{12} there corresponds one extremal of the integral j, which with $C_{12'}$ satisfies the corner condition* III *at* 2'. *The two parameter set of extremals for J thus defines another two parameter set for j with initial points on S. The equations of the two sets with u and v as parameters can be put in the forms* (10) *and* (12)

Furthermore the functional determinant Δ of the functions defined in (12)

$$\Delta(s, u, v) = \begin{vmatrix} \varphi_s & \varphi_u & \varphi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{vmatrix}$$

is found to be different from 0 for the point 2 and hence from continuity conditions for points near 2.

* For a method of evaluating see Bliss and Mason, I, p. 447.

If however it is assumed that Δ does vanish for some point 4 on C_{23} and that at least one of the three-rowed determinants of the matrix

$$\begin{Vmatrix} \Delta_s & \Delta_u & \Delta_v \\ \varphi_s & \varphi_u & \varphi_v \\ \psi_s & \psi_u & \psi_v \\ \chi_s & \chi_u & \chi_v \end{Vmatrix}$$

does not vanish with Δ, it then follows* that the extremals (12) have an enveloping surface D which touches the curve C_{23} at 4. Moreover it is known from the general theory that there is a single definite curve

$$d: \qquad x = x(\alpha), \quad y = y(\alpha), \quad z = z(\alpha)$$

on S which touches C_{23} at 4 and is the envelope of a one parameter family of extremals selected from (12) and containing C_{23} for $\alpha = 0$. The equation of this last family may be written in the form

$$(13) \qquad x = \varphi(s, \alpha), \quad y = \psi(s, \alpha), \quad z = \chi(s, \alpha).$$

Furthermore it is possible to select from the two-parameter family (10) a one-parameter family involving the parameter α and satisfying the direction conditions III. For in connection with these equations it was shown that any $C_{12'}$ determined uniquely a $C_{2'3}$ with initial point on S. Since however the functional determinants for both sets of extremals are different from zero at 2 it follows that the converse also holds. Consequently when a one-parameter family (13) satisfying the imposed conditions is chosen from (12) there goes with it a definite one-parameter family of the set (10), say

$$(14) \qquad X = \Phi(s, \alpha), \quad Y = \Psi(s, \alpha), \quad Z = \mathrm{X}(s, \alpha),$$

containing C_{12} for $\alpha = 0$.

§ 4. *The Jacobi Condition.*

By means of the two one-parameter families just given and their enveloping curve d it can now be proved that an arc C_{123} which joins 1 to 3 and minimizes the sum of the two integrals J and j cannot have upon it a point of contact 4 with

* See Bliss and Mason, I, p. 449.

the enveloping surface D. For consider the sum of the integral J taken along $C_{12'}$ from 1 to S, plus the value of j taken along $C_{2'3'}$ from S to the contact point 4′ of $C_{2'3'}$ with d and then, along d from 4′ to 4. It is found that this sum has a constant value, for its derivative with respect to α is zero.

In fact it is readily verified by the ordinary methods that

$$\frac{dJ}{d\alpha}(C_{12'}) = x_a F_{X'} + y_a F_{Y'} + z_a F_{Z'}/^{2'},$$

where x_a, y_a, z_a are the direction cosines of a line on S. Likewise

$$\frac{dj}{d\alpha}(C_{2'4'}) = -[f_{x'}x_a + f_{y'}y_a + f_{z'}z_a]/^{2'} + f(x, y, z, x_a, y_a, z_a)/^{4'},$$

and

$$\frac{dj(d_{4'4})}{d\alpha} = -f(x, y, z, x_a, y_a, z_a)/^{4'}.$$

Recalling now the direction condition at the point 2′, it is seen that the sum of these three derivatives is zero and therefore the sum of the three integrals $J(C_{12'})$, $j(C_{2'4'})$ and $j(d_{4'4})$ is constant in value and in particular is equal to the sum of $J(C_{12})$ and $j(C_{24})$.

The usual argument with regard to the envelope d not being a solution of the Euler equations can now be applied, from which it follows that if 4 is not a singular point of d then the arcs C_{12} and C_{23} cannot minimize the sum $J_{12} + j_{23}$ if 4 lies on the arc C_{23}. Therefore as a fourth necessary condition it can be stated that

IV. *The extremal arcs C_{12} and C_{23} can contain no points conjugate to their initial points* 1 *and* 2. *Therefore the curve d on the enveloping surface D of the extremals $C_{2'4'}$ must not touch the arc C_{23} before the point* 3.

§ 5. *Sufficient Conditions.*

Suppose now that a curve C_{123} has been found which satisfies conditions I, II, III and IV strengthened by the assumption that: IV′, the curve d does not touch the arc C_{23} even at the point 3. It is desired to see if C_{123} actually minimizes the sum of the two integrals under these conditions.

It has already been shown that if the arc C_{12} does not contain the point conjugate to 1 it can be imbedded in a two-para-

meter family of extremals $C_{12'}$ passing through 1. Moreover the functional determinant of this family is different from zero and it follows therefore that the set $C_{12'}$ forms a field F' about C_{12}. Further, on account of condition IV′, the determinant $\Delta(s, u, v)$ of the set $C_{2'4'}$ determined by $C_{12'}$ and condition III is different from zero along C_{23} and so the set $C_{2'4'}$ forms a field F'' about the arc C_{23}. In each of these fields the properties of the extended invariant integral* hold.

Consider then any curve

$$C: \qquad x = x(t), \quad y = y(t), \quad z = z(t)$$

which joins the two fixed points 5 and 7, crosses S once at 6 and lies entirely in the fields F' and F''. It can be proved that the following theorem holds: *The sum of the two integrals $J^*(C_{56})$ and $j^*(C_{67})$ is independent of the path of integration and depends only on the end points 5 and 7.*

In order to prove this consider any comparison curve $\overline{C}$ which joins 5 and 7, lies entirely in the fields F' and F'', and crosses S once at the point 6′. Since each of the integrals is invariant in its respective field the two following equations result

$$J^*(\overline{C}_{56'}) + J^*(k_{6'6}) = J^*(C_{56}),$$
$$j^*(k_{6'6}) + j^*(\overline{C}_{6'7}) = j^*(C_{67}),$$

where k is any curve on S. Combining the two equations,

$$J^*(\overline{C}_{56'}) + j^*(\overline{C}_{6'7}) - [J^*(\overline{C}_{56}) + j^*(C_{67})]$$
$$= - J^*(k_{6'6}) - j^*(k_{66'}) = J^*(k_{66'}) - j^*(k_{66'}).$$

If expressed in the form of a definite integral, the right hand member of this equation is

$$\int_6^{6'} \{a'F_{x'} + b'F_{y'} + c'F_{z'} - a'f_{x'} - b'f_{y'} - c'f_{z'}\}dt,$$

where a', b', c' are the direction cosines of k. But condition III tells at once that the integrand is identically zero and the above statement follows.

Hence if C_{567} coincides with an extremal for J from 5 to 6 and with an extremal for j from 6 to 7, then

$$J^*(C_{56}) + j^*(C_{67}) = J(C_{56}) + j(C_{67}).$$

* For a statement of these properties see Bliss and Mason, I, p. 458.

Therefore when $\overline{C}$ is any curve in F' and F'' joining the points 1 and 3 and C is the corresponding curve satisfying conditions I, II, III, IV′, the preceding equations give the relation

$$J^*(\overline{C}_{12'}) + j^*(\overline{C}_{2'3}) = J^*(C_{12}) + j^*(C_{23}) = J(C_{12}) + j(C_{23}).$$

This equation may also be written in the form

$$\begin{aligned} J(\overline{C}_{12'}) + j(\overline{C}_{2'3}) - [J(C_{12}) + j(C_{23})] \\ = J(\overline{C}_{12'}) - J^*(\overline{C}_{12'}) + j(\overline{C}_{2'3}) - j^*(\overline{C}_{2'3}) \\ = \int_{\overline{C}_{12'}} E dt + \int_{\overline{C}_{2'3}} e dt, \end{aligned}$$

where E and e are the extended Weierstrass E-functions. On account of II these functions are nowhere negative and therefore

$$J(\overline{C}_{12'}) + j(\overline{C}_{2'3}) > J(C_{12}) + j(C_{23}).$$

Hence the conclusion: *Under the hypothesis imposed the curve C_{123} actually minimizes the sum of the integrals J and j if it satisfies the conditions* I, II, III, IV′.

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THE DEGREE OF A CARTESIAN MULTIPLIER.

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(Read before the Chicago Section of the American Mathematical Society, April 8, 1910, and April 5, 1912.)

1. A large part of Laguerre's numerous and important contributions to the theory of algebraic equations* is based on Descartes' rule of signs, and especially on its application to infinite series. One of the most fertile ideas developed is that an upper limit for the number of real roots of a polynomial with real coefficients, $f(x)$, in an interval $[0, a]$ results from the application of the rule of signs to a product $f_2(x) = f_1(x)f(x)$ developed in a power series which converges for $|x| < a$, but

* See in particular the memoir, "Sur la théorie des équations numériques," Oeuvres, pp. 3–47.

diverges for $x = a$. Thus integral functions $f_1(x)$ exist for which this method gives, in general, a better approximation for the number of positive real roots of $f(x)$ than could be obtained by the direct use of the rule of signs on $f(x)$ itself. In particular Laguerre has shown (loc. cit.) that if the product $e^{zx}f(x)$ is developed according to ascending powers of x, the number of variations of sign in the sequence of the coefficients will be exactly equal to the number of positive real roots of $f(x)$, provided z is a sufficiently large positive number. Recently it has been proved by Fekete and Pólya* that the rule of signs applied to the power series for $f(x)/(1-x)^n$ gives the exact number of real roots of $f(x)$ in the interval $[0, 1]$ if n is sufficiently large, a result easily modified so as to apply to the interval $[0, \infty]$; and in the same paper it is announced that the multiplier $(1+x)^n$, with a similar restriction on n, may also be used to obtain the number of positive real roots.

A question of particular interest is the existence and character for a given polynomial with real coefficients, $f(x)$, of what I have elsewhere† styled Cartesian multipliers, i. e., polynomials $f_1(x)$ such that the rule of signs applied to the product $f_2(x) = f_1(x)f(x)$ gives exactly the number of positive real roots of $f(x)$. A discussion of such multipliers for polynomials having no positive roots has been given by E. Meissner,‡ and in the general case by the present writer in the paper above cited, where their existence is made a corollary of Laguerre's theorem concerning the product $e^{zx}f(x)$. Their existence, of course, also follows from the results obtained by Fekete and Pólya.

In the present paper I shall give a proof of the existence of Cartesian multipliers which is in some ways simpler than any yet given.§ That the construction of a multiplier by the method indicated would require a knowledge of the roots seems an objection, but this is met by a theorem of my *Annalen* paper above referred to, which states that if there is a Cartesian multiplier of degree r, there is always at least one of,

* "Über ein Problem von Laguerre," *Palermo Rendiconti*, vol. 34 (1912), p. 89.

† "An extension of Descartes' rule of signs," *Math. Annalen*, vol. 73 (1912), p. 424.

‡ "Über positive Darstellungen von Polynomen," *Math. Annalen*, vol. 70 (1911), p. 223.

§ This proof possibly antedates the others, having been announced in April, 1910, as noted above.

that degree for which the product function $f_2(x)$ has no more than $n+1$ non-vanishing coefficients, where n is the degree of $f(x)$. The number of product functions of degree $r+n$ with r terms lacking is evidently finite; if we pick out those functions of this set whose coefficients present the minimum number of variations of sign, and divide each by $f(x)$, we obtain Cartesian multipliers whose coefficients are rational in the coefficients of $f(x)$. From this point of view it is important to obtain as low a value for r as possible, and in a form not requiring a knowledge of the roots of $f(x)$. Since $x^k f_1(x)$ is for all positive integral values of k a Cartesian multiplier of $f(x)$ if $f_1(x)$ has this property, it is obvious that there exist Cartesian multipliers of all degrees greater than the minimum r. Although the evaluation for the degree of a Cartesian multiplier obtained in the following pages is far too large, in general, for practical applications, it has at least the interest of a first attempt at an explicit formula.

2. We shall consider first the case where $f(x)$ is a quadratic with imaginary roots. For a very ingenious geometric method the reader should consult the paper of Meissner above cited.

The case of interest is, of course, where $f(x)$ has the form $x^2 - ax + b$ ($a^2 < 4b$), a and b being positive real numbers, this being the only form for which Descartes' rule is inexact. The problem is to assign a multiplier $f_1(x)$ such that the product $f(x)f_1(x)$ will have no negative coefficients.

An especially simple solution is the following: If we multiply $x^2 - ax + b$ by $x^2 + ax + b$, the result will be of the form $x^4 - a_1x^2 + b^2$; if a_1 is negative, we stop here, otherwise we multiply the last expression by $x^4 + a_1x^2 + b^2$, and keep up the process thus indicated. When this has been carried sufficiently far we shall obtain a product of the form $x^{2^{k+1}} - a_kx^{2^k} + b^{2^k}$, where a_k is negative or zero. The truth of this is evident if the roots of $f(x)$ are $\rho e^{i\phi}$, $\rho e^{-i\phi}$, so that $a = 2\rho \cos \phi$ and $b = \rho^2$; it then follows at once that $a_1 = 2\rho^2 \cos 2\phi$, and $a_k = 2\rho^{2^k} \cos 2^k\phi$. Since a is positive, ϕ can be taken as an angle of the first quadrant, and k may be chosen as the least positive integer for which $2^k\phi \geqq \pi/2$.

The value $2^{k+1}-2$ thus obtained as the degree of a Cartesian multiplier for a quadratic of the above form is not, in general, the minimum one. This has been proved by Meissner, in the paper cited, to be the least positive integer r for which $(r+2)\phi \geqq \pi$. The same result has been obtained by the

author (loc. cit.) as a corollary of the theorem that there is a Cartesian multiplier of minimum degree such that the product function is of form $x^{r+2} + Ax^s + B$. The coefficients A and B are easily determined from the fact that the roots of $f(x)$ must be roots of the product; s may be any positive integer less than $r + 2$. The corresponding Cartesian multipliers are easily computed.

Though a solution of our problem in this case has been thus obtained, the interest attaching to Laguerre's multiplier e^{xz} will perhaps warrant a brief digression to examine the Cartesian multiplier formed by taking $m + 1$ terms of its development in powers of x. If the product is arranged in the form

$$c_0 + c_1x + \cdots + c_{m+2}x^{m+2},$$

the coefficients are given by the formulas

$$c_k = \frac{z^{k-2}}{k!}(\rho^2z^2 - 2\rho zk\cos\phi + k(k-1)) \qquad (k \leqq m),$$

$$c_{m+1} = \frac{z^{m-1}}{m!}(-2\rho z\cos\phi + m),$$

$$c_{m+2} = \frac{z^m}{m!}.$$

We can make c_k positive or zero for all values of $k \leqq m$, no matter how large m is taken, if

$$z = \frac{1}{2\rho(1-\cos\phi)},$$

and c_{m+1} will then be positive if

$$m \geqq \frac{\cos\phi}{1-\cos\phi}.$$

If we compare this degree m with r in the preceding paragraph, we see that m is never less than r, and is much larger for small values of ϕ.

3. We shall now form a multiplier for any polynomial $f(x)$ of degree n by a method which depends upon a knowledge of the roots, leaving for the following section the problem of obtaining an upper limit M for the degree of this multiplier in an explicit expression in terms of the coefficients of $f(x)$.

According to § 1 we are then assured of the existence of Cartesian multipliers of all degrees $\geqq M$ whose coefficients are rational in terms of those of $f(x)$.

Let us write our polynomial of degree n in the form

$$f(x) = p_{2n_1}(x)q_{n_2}(x)s_{n_3}^{r}(x), \tag{1}$$

where each of the factors p_{2n_1}, q_{n_2}, s_{n_3}, is a polynomial of the degree indicated by its subscript, the first having imaginary roots only, the second only zero or negative roots, and the last only positive roots. If $n_1 = 0$ the rule of signs for $f(x)$ is exact, hence the case to be considered is that where $n_1 > 0$.

We have just proved that each quadratic factor of p_{2n_1} has a multiplier such that the product has all its terms positive. The product $P(x)$ of these multipliers will then be a Cartesian multiplier for $p_{2n_1}(x)q_{n_2}(x)$, since the coefficients of $q_{n_2}(x)$ are all of like sign. Let us designate the degree of the product $P(x)p_{2n_1}(x)q_{n_2}(x)$ by N. Since we are considering the case where $n_1 > 0$, we must have $N \geqq 2$.

The polynomial $s_{n_3}(x)$ has the factored form

$$s_{n_3}(x) = k\prod_{i=1}^{i=n_3}(x - \alpha_i),$$

where k is a real constant, and the numbers α_i are all real and positive. If this is multiplied by

$$S(x) = \prod_{i=1}^{i=n_3}(x^{N-1} + x^{N-2}\alpha_i + \cdots + x^{N-h}\alpha_i^{h-1} + \cdots + \alpha_i^{N-1}),$$

where N has the value indicated above, we shall have

$$S(x)s_{n_3}(x) = k\prod_{i=1}^{i=n_3}(x^N - \alpha_i^N) = k\sum_{j=0}^{j=n_3}(-1)^j\lambda_j x^{(n_3-j)N},$$

where every λ_j is positive. It is now evident, if $S(x)s_{n_3}(x)$ and $P(x)p_{2n_1}(x)q_{n_2}(x)$ are arranged as polynomials in descending powers of x, that their product will present $n_3 + 1$ sequences of terms such that all coefficients of the same sequence will have the same sign, which will be opposite to that of the coefficients in the succeeding sequence. The product will thus present n_3 variations of sign, where n_3 is the number of positive real roots of $f(x)$. But

$$P(x)p_{2n_1}(x)q_{n_2}(x)\cdot S(x)s_{n_3}(x) = P(x)S(x)f(x),$$

so that $P(x)S(x)$ is a Cartesian multiplier of $f(x)$.

4. It remains to obtain an upper limit for the degree of the multiplier $P(x)S(x)$ described in § 3, and to put this in a form depending explicitly on the coefficients of $f(x)$.

The polynomial $P(x)$ was obtained as the product of Cartesian multipliers for each of the quadratic factors of $p_{2n_1}(x)$. Hence if m_1 is not less than the degree of any of these multipliers, the degree of $P(x)$ will be at most m_1n_1. From its definition, the number N verifies the equation

$$N = m_1n_1 + 2n_1 + n_2,$$

if m_1n_1 is the degree of $P(x)$. Since

$$n = 2n_1 + n_2 + n_3,$$

the above may be written

$$N = m_1n_1 + n - n_3.$$

Hence, the degree of $S(x)$ being $n_3(N-1)$, the degree of $P(x)S(x)$ is at most equal to

$$m = m_1n_1 + n_3(N-1) = m_1n_1(1+n_3) + n_3(n-n_3-1).$$

But

$$2n_1 \leqq n - n_3,$$

so that

$$m \leqq m_1\frac{n-n_3}{2}(1+n_3) + n_3(n-n_3-1).$$

This expression is a quadratic in n_3 whose maximum value, reached when $n_3 = (n-1)/2$, is easily obtained. In case this is an integer, it will serve for M, or, more generally,

$$M = E\left[\left(\frac{m_1}{2}+1\right)\left(\frac{n-1}{4}\right)^2 + \frac{m_1}{2}n\right], \tag{2}$$

where the symbol $E(a)$ denotes the largest integer not greater than the real number a.

This number M will be the degree of a Cartesian multiplier in all cases. To be sure, it was obtained under the hypothesis $n_1 > 0$, but as before observed, if $n_1 = 0$ the rule of signs is exact for $f(x)$ itself, so that x^M, for instance, will be a multiplier of degree M.

We now proceed to express the number m_1 in the above

formula in terms of the coefficients of $f(x)$. This number, it will be recalled, is a positive integer large enough to be the degree of a Cartesian multiplier for any quadratic factor of $f(x)$ corresponding to a pair of conjugate imaginary roots.

We have seen in § 2 that a quadratic

$$x^2 - 2\rho \cos \phi \cdot x + \rho^2$$

has a multiplier of degree r, where r is the least positive integer such that

$$(r + 2)\phi \geqq \pi,$$

and there will be multipliers of any degree greater than that given by this formula. It follows that if ψ is a positive angle less than all the positive arguments of the imaginary roots of $f(x)$, we can take for m_1 the least positive integer verifying the inequality

$$(m_1 + 2)\psi \geqq \pi. \tag{3}$$

A simple method for obtaining an evaluation for ψ in terms of the coefficients of $f(x)$ is suggested by Cauchy's treatment of the problem of determining a minimum absolute value for the differences of real roots of a polynomial.*

Since the removing of multiple roots from $f(x)$ involves only rational processes, we will here suppose that they are absent. Let the roots of $f(x)$, all distinct, be designated by $x_1, x_2, \cdots, x_n$, and the discriminant of $f(x)$ by D. Then, by a well-known formula†

$$D = (-1)^{\frac{n(n-1)}{2}} a_0^{2n-2} \prod_{\lambda, \mu} (x_\lambda - x_\mu)^2$$

$$(\lambda = 1, 2, \cdots, n-1;\ \mu = \lambda + 1, \lambda + 2, \cdots, n).$$

If the minimum positive argument of imaginary roots is ϕ_1, the corresponding pair of roots being $\rho_1 e^{\pm i\phi_1}$, and if ρ denotes a superior limit for the moduli of roots of $f(x)$,‡ we have, since $|x_\lambda - x_\mu| \leqq 2\rho$,

$$|D| \leqq |a_0|^{2n-2}(2\rho_1 \sin \phi_1)^2(2\rho)^{n(n-1)-2}$$

$$\leqq |a_0|^{2n-2}(2\rho)^{n(n-1)}(\sin \phi_1)^2,$$

or

* Netto, Vorlesungen über Algebra, vol. 1, p. 272.

† Netto, l. c., p. 177.

‡ Such a limit is, for example, given by the formula $\rho = 1 + r/r_0$, where $r_0 = |a_0|$, and r is the greatest of the numbers $|a_0|, |a_1|, \cdots, |a_n|$.

$$\sin \phi_1 \geqq \frac{|D|^{\frac{1}{2}}}{|a_0|^{n-1}(2\rho)^{n(n-1)/2}}.$$

Solved for ϕ_1, this gives

$$\phi_1 \geqq \sin^{-1} \frac{|D|^{\frac{1}{2}}}{|a_0|^{n-1}(2\rho)^{n(n-1)/2}}. \tag{4}$$

Since $\phi_1 \geqq \sin \phi_1$, (4) may be replaced by

$$\phi_1 \geqq \frac{|D|^{\frac{1}{2}}}{|a_0|^{n-1}(2\rho)^{n(n-1)/2}}. \tag{5}$$

If the coefficients of $f(x)$ are integers, D is a whole number, and $|D| \geqq 1$, so that (4) and (5) can in this case be replaced by the stronger but simpler inequality

$$\phi_1 \geqq \frac{1}{|a_0|^{n-1}(2\rho)^{n(n-1)/2}}. \tag{6}$$

We may take for ψ the right-hand side of (4) or (5) or, in the case indicated, of (6).

The problem of obtaining the degree of a Cartesian multiplier is thus solved by formulas (2) *and* (3), ψ *being taken equal to the right-hand member of either* (4), (5), *or* (6).

It must be confessed that the evaluation thus obtained is absurdly high, even in simple cases. Thus, to take an example somewhat at random, if $f(x) = x^4 + x^3 - 2x^2 + 2x + 3$, the least value of m_1 as given by (3) and (4) is not far from 8,500, and (2) would give M as greater than 19,000. I have shown in my *Annalen* paper that this polynomial has a Cartesian multiplier of degree 2. It is not likely that the methods here given, even if improved, will have much value for numerical computation. There remains, however, the interest attaching to a formal solution of the problem in hand.

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ON CLOSED CONTINUOUS CURVES.

BY PROFESSOR ARNOLD EMCH.

(Read before the Southwestern Section of the American Mathematical Society, November 30, 1912.)

1. In a paper soon to be published it will be proved that in every closed convex curve which is analytic throughout at least one square may be inscribed. Conversely, if in a Cartesian plane an arbitrary square is given, the problem is to find the parametric equations of any convex curve through the vertices of the given square. It is the purpose of this paper to establish these equations.

2. First assume the square $A_1A_2A_3A_4$ symmetric with respect to the x- and y-axes and on the circle

$$(1) \qquad x = \frac{a}{2}\sqrt{2}\cos\frac{2\pi t}{w}, \qquad y = \frac{a}{2}\sqrt{2}\sin\frac{2\pi t}{w},$$

where t is the parameter and w the period. The coordinates of A_1, A_2, A_3, A_4 are in the same order $(a/2, a/2)$; $(-a/2, a/2)$; $(-a/2, -a/2)$; $(a/2, -a/2)$, and the corresponding parameters $t_k = (2k+1)\cdot w/8$, $(k = 0, 1, 2, 3)$. Designating by $\phi(t)$, $\psi(t)$ two uniform continuous functions for all values of t, and with the same period w, then

$$(2) \qquad \begin{aligned} x &= \frac{a}{2}\sqrt{2}\cos\frac{2\pi t}{w} + \lambda\sin\left(\frac{2\pi t}{w} - \frac{\pi}{4}\right)\sin\left(\frac{2\pi t}{w} - \frac{3\pi}{4}\right)\phi(t), \\ y &= \frac{a}{2}\sqrt{2}\sin\frac{2\pi t}{w} + \mu\sin\left(\frac{2\pi t}{w} - \frac{\pi}{4}\right)\sin\left(\frac{2\pi t}{w} - \frac{3\pi}{4}\right)\psi(t),* \end{aligned}$$

where λ and μ are arbitrary constants, represent a closed curve through the vertices of the square. Now any closed continuous curve† may be represented parametrically by

$$(3) \qquad x = F(t), \qquad y = G(t),$$

* In this expression it is sufficient to take the product of two sines as indicated, not as in the abstract of the paper which appeared in the BULLETIN of February, pp. 221-222, where a product of four sines was introduced.

† Osgood: Lehrbuch der Funktionentheorie, vol. 1, 2d ed., pp. 146–150.

in which F and G are of the same general type as ϕ and ψ. Hence any closed curve through $A_1A_2A_3A_4$ may be represented in the form (3). We simply have to choose $F(t)$ and $G(t)$ in such a manner that for $t = (2k+1)\cdot w/8$, $k = 0, 1, 2, 3$, they assume the same values as x and y in (1) and (2). Supposing that $F(t)$ and $G(t)$ satisfy this condition, it must be shown that $\phi(t)$ and $\psi(t)$ in (2) can be determined as continuous coperiodic functions of t, defined for all values of t, such that the right-hand members of (2) become respectively $F(t)$ and $G(t)$. Evidently for all values of t, except $t \equiv (2k+1)\cdot w/8 (\text{mod } w)$,

$$(4)\qquad \phi(t) = \frac{F(t) - \frac{a}{2}\sqrt{2}\cos\frac{2\pi t}{w}}{\lambda \sin\left(\frac{2\pi t}{w} - \frac{\pi}{4}\right)\sin\left(\frac{2\pi t}{w} - \frac{3\pi}{4}\right)},$$

$$\psi(t) = \frac{G(t) - \frac{a}{2}\sqrt{2}\sin\frac{2\pi t}{w}}{\mu \sin\left(\frac{2\pi t}{w} - \frac{\pi}{4}\right)\sin\left(\frac{2\pi t}{w} - \frac{3\pi}{4}\right)}.$$

For all values of t, different from odd multiples of $w/8$, $\phi(t)$ and $\psi(t)$ as defined by (4) are continuous and well defined. For $t = (2k+1)\cdot w/8$ both $\phi(t)$ and $\psi(t)$ become indeterminate. We find, however,

$$\lim_{t=(2k+1)w/8}\{\phi(t)\} = \lim_{t=(2k+1)w/8}\left\{\frac{F'(t) + \frac{a\sqrt{2}\cdot\pi}{w}\sin\frac{2\pi t}{w}}{\frac{\lambda 2\pi}{w}\sin\left(\frac{4\pi t}{w} - \pi\right)}\right\}$$

$$= \frac{F'(t) + \frac{a\pi\sqrt{2}}{w}\sin(2k+1)\frac{\pi}{4}}{\frac{\lambda 2\pi}{w}\sin(2k-1)\frac{\pi}{2}}$$

$$= \pm\frac{wF'(t) \pm a\pi}{2\lambda\pi},$$

which is evidently a finite quantity. The same is true of $\lim\{\psi(t)\}$ for the same values of t.

From this follows that $\phi(t)$ and $\psi(t)$ as defined by (4) are continuous for all values of t and, consequently, that any

closed analytic curve through the vertices of the given square may be parametrically represented by (2).

3. Applying to the points (x, y) of the Cartesian plane a combined rotation and translation θ, p, q, so that after the motion the coordinates of the displaced points with respect to the same plane are

$$\begin{aligned} x' &= p + x\cos\theta - y\sin\theta, \\ y' &= q + x\sin\theta + y\cos\theta, \end{aligned} \tag{5}$$

and choosing the side a of the square properly, the transformed square $A_1'A_2'A_3'A_4'$ may represent any square in the plane. Putting

$$\begin{aligned} \lambda\phi(t)\cos\theta - \mu\psi(t)\sin\theta &= f(t), \\ \lambda\phi(t)\sin\theta + \mu\psi(t)\cos\theta &= g(t), \end{aligned} \tag{6}$$

the parametric equations of the closed curve through the square in the new position, after substituting in (5) for x and y their expressions as given in (2), and reducing, may be written

$$\begin{aligned} x' = p + \frac{a}{2}\sqrt{2}\cos\left(\frac{2\pi t}{w} + \theta\right) \\ + \sin\left(\frac{2\pi t}{w} - \frac{\pi}{4}\right)\sin\left(\frac{2\pi t}{w} - \frac{3\pi}{4}\right)f(t), \\ y' = q + \frac{a}{2}\sqrt{2}\sin\left(\frac{2\pi t}{w} + \theta\right) \\ + \sin\left(\frac{2\pi t}{w} - \frac{\pi}{4}\right)\sin\left(\frac{2\pi t}{w} - \frac{3\pi}{4}\right)g(t). \end{aligned} \tag{7}$$

In (6) $\phi(t)$ and $\psi(t)$ may always be determined in such a manner that $f(t)$ and $g(t)$ are any two distinct continuous and co-periodic functions of t.

A closed analytic curve through the vertices of a square may therefore always be represented by the parametric equations (7).

As it is always possible to inscribe at least one square in an analytic convex curve* (ordinary oval), the parametric equations of such a curve may always be written in the form (7).

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* The proof for this theorem will appear in the *American Journal of Mathematics*. Recently I have been able to prove that the theorem holds for any closed analytic curve without singular points.

LET US HAVE OUR CALCULUS EARLY.

The Calculus for Beginners. By J. W. MERCER. Cambridge (England), University Press, 1910. xiv + 440 pp.

THE decline of the Græco-Roman empire over our collegiate studies has been most marked and is apparently extending almost to extinction. Many reasons may be assigned for this fall. One is the widening range of knowledge and interest, another the passing of the professional and utilitarian aspects of Greek and Latin, another the failure of these subjects to make good.

We must not forget that our colleges were for the most part started by ministers of the gospel, and chiefly as divinity schools. In the early days, even if not to such an extent at present, Greek and Latin were just as utilitarian and necessary professional subjects for the young student of divinity as mathematics now is for the electrical engineer and the physicist. The talk of their value as cultural and disciplinary studies is probably of later and comparatively recent date, an invention of those vast vested interests who would delay, even though they cannot stay, the march of progress away from them. The percentage of our present undergraduate collegians for whom the ancient languages are utilitarian and professional is very small.

That culture and valuable intellectual discipline are best obtained by application to subjects which are neither useful nor interesting to the student, and over which he never obtains even a mediocre mastery, is an idea which is losing ground and must necessarily lose ground despite the extent and intrenchment of the aforesaid vested interests. The fact is that Greek and Latin do not make good. After studying them for six to eight years the student cannot read them with ease or profit; they form but a comparatively small part of his mental outfit, whether for pleasure or work.

If the classics were still of widespread professional necessity, or if the study of them throughout the preparatory course and for two or three years in college gave a pleasurable mastery over them, it is far from likely that they would occupy their present low estate even in the face of the widening range of knowledge, the transfer of executive interest in our colleges

from accomplishment to equipment and from high standards to numbers, and the accompanying transition of student interest from study to sport.

We mathematicians, however, are in no position to gaze upon the motes in the eyes of our classical brethren, to whom we can hardly cómpare ourselves favorably. For there has been a great decline in the sway of mathematics over collegiate education. We cannot attribute this decline nearly so much as in the case of classics to a passing of professional or utilitarian interest. Fifty years ago who needed mathematics? Those who were to teach it, a few theoretical astronomers, some terrestrial surveyors. Now, added to these, there is the vast host of engineers, civil, mechanical, electrical, and others, who should be familiar with the calculus. We cannot so well attribute the decline to the widening range of knowledge; for to some of the newer subjects, notably physics, mathematics is complementary rather than competitive. We suffer, of course, in common with all the severer disciplines, by the introduction of large numbers of "cultural" and "snap" courses, by the presence in college of great numbers of fellows neither primarily nor seriously there for the sake of intellectual advancement. But our chief difficulty is that we do not make good.

By not making good is meant that after a long course in preparatory mathematics and a couple of years in college most students, indeed all except the most extraordinary, are unable to use their mathematics, that is, they find their mathematics, especially that learned in college, an ineffective part of their mental outfit whether for pleasure or work. Were it not for a fancied unique training, more imaginary than real, that mathematics is supposed to give, were it not for the obvious professional necessities of the engineer, we should find collegiate mathematics worse off than classics, and most of us mathematicians without a job. With a full realization of the fact that our students cannot get out of our courses as much as we may wish them to get, with all proper disdain for those practical persons who desire the student to acquire mathematics merely as a tool—an impossibility—, those of us who are honest and not too conceited cannot fail to feel that we do not make good. As external evidence of this fact we need only consider the large number of texts (by incompetent mathematicians) now appearing under such titles

as Mathematics for the Chemist, Engineering Mathematics, and so on.

One of the main troubles with us is that we do not select the right subjects to teach in the early collegiate years.* There is no sense in giving the freshman a considerable course on advanced algebra. The subject is abstract and deals with topics and ideas relatively unimportant for the student. Yet advanced algebra is often taught as a prerequisite to calculus. It is unfortunate to force the freshman through an extensive course in analytic geometry. So much of the subject as deals with rectangular (and possibly polar) coordinates and with the tracing of simple curves is indeed necessary, and is closely related to methods in use in a wide variety of everyday studies. This much we may call graphical representation. But the real essence of analytic geometry, the obtaining of geometric results by algebraic means, is of small use save to the pure mathematician and is rarely comprehended by the immature student.

That mathematical subject which is most vitally important for the general student is the calculus, differential and integral. It is no less important for the infrequent future mathematician. It is the calculus which should form the greatest possible portion of the ordinary freshman course in mathematics. When the calculus is postponed until the sophomore year and follows somewhat elaborate courses on trigonometry, advanced algebra, or analytic geometry, it can be made so abstract, so formal, so purely mathematical as to leave little impression upon any save the best students. This is the chief danger in postponing it. If the teacher can rely upon only a very meager preparation, he has to keep the analysis elementary to the point where it offers small difficulty even to the student who is not particularly facile with his algebra, he is able to give his time to a variety of simple but fundamental applications which rivet the attention of the student, he has a chance, which he otherwise does not have, to make good.

Such considerations as these are probably at the back of the

* In an engineering school where the student is required to take mathematics for a definite period the order and treatment of topics does not need to be so tempered to the shorn lamb as in a college where there is a tolerably free range of election, where many more will take one year of mathematics than two, and where the conduct of the first year is therefore of prime importance both for the general good and for subsequent elections in mathematics.

somewhat general, and the highly commendable, tendency which we are now witnessing toward the earlier introduction of the calculus. This is undoubtedly the idea which has inspired The Calculus for Beginners by J. W. Mercer, head of the mathematical department at the Royal Naval College, Dartmouth, England. The author states in his preface: "I have been guided by my conviction that it is much more important for the beginner to understand clearly what the processes of the calculus mean, and what it can do for him, than to acquire facility in performing its operations or a wide acquaintance with them. I had much rather that a boy, confronted with a problem, should, after analyzing it, be able to say 'If I could differentiate (or integrate) this function of x, I could solve the problem,' than that he should be able to perform the operation without seeing its bearing on the problem. . . . A boy is supposed to know his elementary algebra and trigonometry and to have some slight acquaintance with the coordinate geometry of the straight line. He should be able to write down the equation of a line through a given point with a given gradient and should know the relation between the gradients of perpendicular lines. He is also expected to have had some practice in drawing graphs from their equations and to know what these graphs mean. . . . The subject has been taught here during the last few years to boys of 16 on the lines of this book. . . ."

Mercer must be a careful teacher; he explains everything very clearly and with great detail, indeed some would say with too great detail. There is no reason why he or any one else should fail in teaching this book to boys of sixteen. Except for the definition of the tangent of an angle no trigonometry is required before page 261, which is two thirds of the way through the book (exclusive of answers and index). To this point the analysis involves merely polynomials in powers (not necessarily integral) of the variable; yet all the primary ideas of differential and integral calculus have been presented, and all the usual applications to maxima and minima, areas, volumes, moments of inertia, and the like, have been given. The requisite amount of coordinate geometry could be given in ten lessons surely, and probably could be explained incidentally to the study of the text without even those preliminary ten lessons.

The exercises set for the student are exceedingy varied and

numerous.* They are distributed a few at a time throughout the text as each new principle is explained. An especially valuable feature, however, is found in the long collections of miscellaneous exercises after Chapters VI, VIII, and XI, which allow the student to be tried out in review upon the chief sections of work, namely, the differential calculus of x^n, the integral calculus of x^n ($n \neq -1$), and the differential and integral calculus of the other elementary functions. It would be safe to say that the student who has been led carefully through the first 260 pages of the text, although being restricted to using only polynomials in x^n, would have a better grasp of the calculus, whether for pleasure or for profit, than he could acquire from the ordinary run of texts on calculus. This ground could probably be covered convincingly in about sixty lessons, and with freshmen.

Chapter I treats uniform speed, average speed, speed at an instant, rate of increase, function, uniform gradient, average gradient, gradient at a point, tangent to a curve, rate of increase as a gradient, test of approximate equality, ratio of continually diminishing quantities. The last two sections give a very clear idea of infinitesimals without the introduction of technical terms. The second chapter, called differentiation from first principles, treats of ds/dt, its connection with $\Delta s/\Delta t$, of dy/dx as a rate of increase, as the ratio of time rates of increase of x and of y, as a gradient, of the significance of the sign of dy/dx. Observe that the author uses speed or rate as primary concept instead of gradient. This is a good thing. Speed and rate are more concrete than slope. The student should learn to think in physical terms, to consider a curve as a convenience in representing physical phenomena, not as an end in itself. The great trouble with students who know something about curves and analytic geometry is that they try to use too much of this knowledge. To them a circular plate suggests coordinate axes and the equation $x^2 + y^2 = a^2$, whereas in most problems in the application of calculus these are entirely superfluous.

After these preliminaries, which fill 74 pages, we come to the differentiation of x^n, to maxima and minima, to small errors and approximations, and to the inverse of differentiation, all of which occupies 78 pages, and is followed by a collection of 50

* The exercises have also been reprinted and published separately from the text.

review exercises. It is interesting to observe that the author treats some problems in maxima and minima where he has to use such a device as finding the maximum or minimum of the square or reciprocal of the given function, because the differentiation would otherwise be impossible (unless obtained from first principles) with the formal methods thus far developed. Some would consider this unfortunate. But it seems to us one of the most fortunate things that can arise. Ingenuity is a very desirable trait to cultivate. The advantage of studying calculus with small preparation and with restricted formal development is precisely that there is wide play for that sort of ingenuity which the student who will later use his calculus is most likely to need. As a matter of fact unless an engineer can squeeze out a solution of his problem by simple analysis alone, he is not likely to trouble to obtain a solution himself.*

Chapter VII is on the integral as a limit of a sum, areas of curves, mean or average ordinate. Simpson's rule is explained and applied. The author nowhere shuns a reasonable amount of calculation, and this is one of the elements that makes his work concrete, practical, and full of meaning. In the eighth chapter further applications are taken up: finding the distance when the speed is given, calculating the work done by a force, finding fluid pressures, volumes of revolution, moments of inertia, centers of gravity, and centers of pressure. Here follows another set of 50 miscellaneous exercises.

The title of Chapter IX is the differentiation of trigonometrical ratios, but we observe that the integration of simple trigonometric expressions is taken up simultaneously. This is as it should be. The exercises are well chosen so far as they go, but it would have been better if they had been more numerous and decidedly more varied. The chapter on polar coordinates (the fourteenth and last in the book) might well have been introduced at this point, furnished with some really satisfactory exercises, and elaborated to include the calculation of moments of inertia, centers of gravity, and the like, of such important figures as yield more easily to treatment by polar

* The idea that an engineer should have in his course only this minimum simple analysis which he is likely himself to use is entirely mistaken; he should have far more. Indeed any student who has such an excellent freshman course as the first eight chapters of this book would make, should then have a systematic drill in formal analysis, including differential equations, with varied applications among which should be found considerable mechanics. This would leave him at the end of his second collegiate year with a very effective knowledge of mathematics.

than by rectangular coordinates. It would have been an excellent review and elaboration upon all the previous ideas in the text.

In Chapter X we find the differentiation of the product, quotient, and function of a function, of inverse and of implicit functions. This is pure formal differentiation, with a trifle of integration, and relatively few applications. In the following chapter we come to the differentiation of n^x, and a general treatment of the exponential and logarithmic function, including simple integrations, the compound interest law, and simple differential equations. The treatment of e is excellent. The author computes with the aid of tables the derivatives of 2^x, 3^x, 2.5^x, and concludes that there must be a number e such that if $y = e^x$, then $dy/dx = y$. The student checks this by differentiating with the tables the function 2.7183^x. The chapter is followed by a collection of 100 miscellaneous exercises.

Chapter XII contains the approximate solution of equations (Newton's method). In Chapter XIII methods of integration by substitution and by parts are given, and the work closes (except for answers and index) with the previously mentioned unsatisfactory chapter on polar coordinates.

We consider that the author would have improved his book had he placed the chapter on the exponential and logarithmic function before Chapter X on the rules of formal differentiation; indeed it might have been still better to have placed this chapter before that on the trigonometric ratios, and to have given more attention to problems requiring statement in a simple differential equation integrable by logarithmic or exponential functions. Such equations arise very frequently in the applications. This, however, is a small matter.

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SHORTER NOTICE.

Introduction to Analytical Mechanics. By ALEXANDER ZIWET and PETER FIELD. Macmillan and Company, New York. ix + 379 pp.

THE book is about of the size and character of Bowser's well-known text. It is based to a large extent on the senior author's Theoretical Mechanics, but the applications to engineering are omitted and the analytical treatment has been broadened. A general knowledge of differential and integral calculus and solid analytics is assumed; the reader is gradually introduced into the subject matter of differential equations and vector analysis but the authors have not deemed it wise to use the symbolic vector notation. The excellence of Professor Ziwet's texts is a matter of common knowledge and that before us is no exception. Likewise the publishers have maintained their established reputation as experts in the way of producing college and university text-books.

The contents of the work can be very well described by: Mechanics of a particle and of the solid body. These two parts have not been kept separate from each other; instead, they are spread out under the three headings of kinematics, statics, and kinetics, so that the reader who follows the text successively will at a rather early period of his study be introduced into the kinematics of the solid body. From a pedagogical standpoint this seems less desirable, since the reader is then generally not mature enough for this subject. Still experience with the book has shown that a teacher may well change the order of material to suit his own convictions.

In the mechanics of a particle the authors have given to part 1, Kinematics, a larger space than one would generally allot to it. There does not seem to be any cogent reason to list the material of pages 42 to 72 under Kinematics instead of Kinetics where it really belongs. Most likely this was done in the recognition that, for the introduction into the kinematics of the solid body, the reader's mind required a wider familiarity with mechanical problems. The examples are generally well chosen and sufficiently easy to make the student grasp the significance of the theory. They are decidedly

easier than some of those in Jeans' Theoretical Mechanics. A gradation between the two classes of problems should be effected by the introduction of illustrative examples which gradually increase in difficulty and to which hints are appended to lead the student to the right solution. We may never be able to attain the proverbial capacity of the English student for solving difficult problems, still we need not rest satisfied with being distanced by them as we are at present. The authors have in general refrained from giving references to standard works in mechanics. The only footnotes appear to be those on pages 256 and 258, the first of which refers to Professor Bôcher's Integral Equations, whilst the second calls attention to Professor Bolza's Calculus of Variations in Teubner's edition. It is difficult to see the good of these references in a text which is written for beginners. A much more appropriate reference would seem to have been Mach's Science of Mechanics or Pearson's Grammar of Science. The finished form of Kirchhoff's Mechanics appeals to us today less than it did thirty years ago, and even an elementary text should well take cognizance of the changed attitude towards the foundations of mechanics since Mach's fundamental contributions.

The mechanics of the solid body has received special attention and care from the authors. The geometrical discussion precedes the analytical treatment and prepares the way for a better understanding of this rather difficult subject. A short account of the fundamental concepts of vector analysis would have been most welcome. The introduction of the symbolic vector notation would have helped the understanding. The remarkable success of Föppl's texts on mechanics, which were originally written for engineering students, shows an advance in this direction which, a decade or two ago, could hardly have been expected. The derivation of Euler's equations and the presentation of the material leading to them are extremely clear and elegant. The authors might well have gone a step further in their presentation and introduced the reader to the concept of elliptic functions. There is hardly a better opportunity for this than in mechanics, since we are used to have the dependent variable in terms of the independent and not the reverse. The careless reader requires special attention to have this fact over and again pointed out to him. A graphic demonstration of the amplitude function is so easily

brought to the understanding of even a dull student that all elementary texts on mechanics presupposing the calculus should not omit this important finishing touch.

A small list of misprints which have come to the attention of the reviewer is herewith given.

P. 27, line 12 from below: omit index in v.
P. 37, line 5 from above: read acceleration times Δt.
P. 40, line 7 from below: read $r\ddot{\theta}$ instead of $r\theta$.
P. 44, line 3 from below: read $v_0 \sin(\epsilon t)$.
P. 56, line 3 from above: read $a(\cos\epsilon\cos\mu t + \sin\epsilon\sin\mu t)$.
P. 61, line 2 from above: read $= OP_x$.
P. 200, line 4 from above: read the potential is constant.
P. 256, line 6 from above: read $\sqrt{k/z}$.
P. 324, line 11 from below: factor dt left out on right side.
P. 361, Article 7, problem 7: 47 miles.
P. 373, Article 56, problem 7: first inside bracket is to be squared.
P. 373, Article 334, problem 7: 44° 34′.

KURT LAVES.

NOTES.

THE July number (volume 14, number 3) of the *Transactions of the American Mathematical Society* contains the following papers: "Proof of the finiteness of modular covariants," by L. E. DICKSON; "On transcendentally transcendental functions," by R. D. CARMICHAEL; "Sur les classes V normales," by M. FRÉCHET; "Implicit functions defined by equations with vanishing Jacobian," by G. R. CLEMENTS; "On the approximate representation of an indefinite integral and the degree of convergence of related Fourier's series," by D. JACKSON; "Certain continuous deformations of surfaces applicable to the quadrics," by L. P. EISENHART.

THE July number (volume 35, number 3) of the *American Journal of Mathematics* contains the following papers: "The primitive groups of class twelve," by W. A. MANNING; "The cartesian oval and the elliptic functions p and σ," by CLARA L. BACON; "The indices of permutations and the derivation therefrom of functions of a single variable associated with the permutations of any assemblage of objects," by P. A. MACMAHON; "Conjugate line congruences of the third order defined by a family of quadrics," by HELEN B. OWENS.

UNDER the auspices of the Edinburgh mathematical society a mathematical colloquium was held at Edinburgh during the week August 4–9. Courses of five lectures each were given by Professor A. W. CONWAY on "The theory of relativity and the new physical ideas of space and time"; by Professor D. M. Y. SOMERVILLE on "Non-euclidean geometry and the foundations of geometry"; and by Professor E. T. WHITTAKER on "Practical harmonic analysis and periodogram analysis; an illustration of mathematical laboratory practice."

THE firm of Martin Schilling in Leipzig announces three new sets of models:

By Professor M. WIEN, Jena: Series XLI, no. 2. Two pendula connected by an elastic spring. Price M 200. No. 3. Two pendula, one suspended on the other. Price M 46.

By Dr. W. KÖNIG, Giessen. Series XLI, no. 4. Thread model to illustrate astigmatism. Price M 75.

By Dr. H. SCHROEDER, Reformrealgymnasium, Halle. Series XLIII, nos. 1–7. Central surfaces of the quadric surfaces: elliptic paraboloid (three models); hyperbolic paraboloid (one model); ellipsoid (three models). Price M 110.

AT a recent session of the Paris academy of sciences the following prizes were awarded.

Poncelet prize (fr. 3000) to Professor MAURICE LEBLANC, for the totality of his researches in mechanics. Pontécoulant prize (fr. 1500) to Dr. SÜNDMAN for his contribution to the problem of three bodies. Binoux prize (fr. 2000) to Professor J. MOLK, of the University of Nancy, for his services to the French edition of the encyclopedia of mathematical sciences.

IN connection with the sixth international congress of mathematicians, to be held in Stockholm in 1916, King Gustav V of Sweden has founded a prize, consisting of a gold medal bearing a portrait of Weierstrass and a cash sum of 3000 crowns, for the best contribution to the theory of analytic functions. Either the general theory of analytic functions or any special branch of important analytic functions may be treated. Competing memoirs should be written in French, German, or English, and sent, under a pseudonym and motto, to the editors of the *Acta Mathematica* by October 31, 1915, the centenary of the birth of Weierstrass. All manuscripts

will be submitted to a jury consisting of the members of the first class of the Swedish academy of sciences, namely, Professors Mittag-Leffler, Falk, Phragmén, Wiman, Bendixson and von Kock, and Professor Fredholm by invitation.

The prize-winning memoir and others of high merit will be published in the *Acta*, and must not be published elsewhere.

THE Adams prize for 1914 will be awarded to the author of the best essay on the following theme:

"The phenomena of the disturbed motion of fluids, including the resistances encountered by bodies moving through them."

The competition is open to any graduate of Cambridge University. The prize is £220.

ON May 25, 1913, Professor FELIX KLEIN, of the University of Göttingen, was presented by his former pupils with a portrait of himself and with a congratulatory letter. The portrait, painted by Max Liebermann, will appropriately adorn the wall of the mathematical institute of the University as soon as the building is completed.

PLANS have been completed for publishing the complete works of the late Henri Poincaré. The publication will be undertaken at once by Gauthier-Villars under the direction of the French minister of public instruction and the academy of sciences of Paris.

OWING to the mass of new material which has been found at St. Petersburg and at other places, the Euler commission realizes that it must face a deficit in the publication of Euler's works, unless further funds are provided. The publication of this new matter will necessitate several additional volumes and involve an unforeseen expense of at least $40,000. To defray this expense it is proposed to form an Euler society, with dues of ten francs per year, the receipts of which are to be devoted entirely to this purpose.

THE annual list of American doctorates published in *Science* presents, for the academic year 1912–1913, 461 names, of which 231 are credited to the sciences. The following 19 successful candidates offered mathematics as major subject (the titles of the theses are appended): D. F. BARROW,

Harvard, "Oriented circles in space"; H. BATEMAN, Johns Hopkins, "The quartic curve and its inscribed configurations"; E. T. BELL, Columbia, "The cyclotomic quinary quintic"; Miss J. E. BURNS, Illinois, "The abstract definitions of the groups of degree eight"; G. R. CLEMENTS, Harvard, "Implicit functions defined by equations with vanishing Jacobian"; C. W. COBB, Michigan, "The asymptotic development for a certain integral function of zero order"; Miss L. P. COPELAND, Pennsylvania, "On the theory of invariants of plane n-lines"; W. A. COIT, Boston, "Introduction to modern geometry"; W. H. CRAMBLET, Yale, "On intermediate functions, being an extension of semi-continuous or upper and lower functions to a classification of discontinuous functions"; G. M. GREEN, Columbia, "Projective differential geometry of triple systems of surfaces"; R. A. JOHNSON, Harvard, "An analytic treatment of the conic as an element of space of three dimensions"; Miss F. P. LEWIS, Johns Hopkins, "A geometrical application of the theory of the binary quintic"; C. E. LOVE, Michigan, "The asymptotic solutions of linear differential equations"; Miss M. L. SANDERSON, Chicago, "Formal modular invariants with an application to binary modular covariants"; J. SLEPIAN, Harvard, "On the functions of a complex variable defined by an ordinary differential equation of the first order and the first degree"; L. L. SMAIL, Columbia, "Some generalizations in the theory of summable divergent series"; W. H. STONE, Boston, "The elements of harmonic ratio"; L. E. WEAR, Johns Hopkins, "On self-dual plane curves of the fourth order"; K. P. WILLIAMS, Princeton, "The solutions of non-homogeneous linear difference equations and their asymptotic form."

THE following advanced courses in mathematics (elementary courses not included) are offered at the Italian universities during the academic year 1913–1914:

UNIVERSITY OF BOLOGNA.—By Professor P. BURGATTI: Mathematical theory of elasticity, three hours.—By Professor L. DONATI: Thermodynamics in its relation to the electromagnetic field and especially to radiations, three hours.—By Professor F. ENRIQUES: Theory of algebraic functions, three hours.—By Professor L. PINCHERLE: Theory of functions of a real variable, Lebesgue's integral, the existence theorems;

elementary theory of the functions of a complex variable with applications to algebraic functions and their integrals, three hours.

UNIVERSITY OF CATANIA.—By Professor E. DANIELE: Electromagnetic field with special regard to the production of energy, three hours.—By Professor M. DE FRANCHIS: Geometry on algebraic curves from the transcendental point of view, four hours.—By Professor G. PENNACCHIETTI: Hydrodynamics, four hours.—By Professor C. SEVERINI: Advanced part of calculus, integral and integro-differential equations, four hours.

UNIVERSITY OF GENOA.—By Professor E. E. LEVI: Differential and integral equations, four hours.—By Professor G. LORIA: Pure synthetic geometry, three hours.—By Professor O. TEDONE: Selected chapters from the theories of potential and of integration of Laplace's equation, three hours.

UNIVERSITY OF NAPLES.—By Professor F. AMODEO: History of mathematics: The epoch of Newton and Leibniz, three hours.—By Professor A. DELRE: n-dimensional analysis of Grassmann with applications to geometry and mechanics, four and one-half hours.—By Professor R. MARCOLONGO: Analytic mechanics: algebraic integrals; problem of three bodies, three hours.—By Professor D. MONTESANO: Linear systems of surfaces; birational correspondences in space, four and one-half hours.—By Professor E. PASCAL: Selected chapters of analysis, differential equations, three hours.—By Professor L. PINTO: Propagation of heat, four and one-half hours.

UNIVERSITY OF PADUA.—By Professor F. D'ARCAIS: Functions of a complex variable, calculus of variations, four hours.—By Professor P. GAZZANIGA: Theory of numbers, three hours.—By Professor T. LEVI-CIVITA: Kinetic and statistic theories with application to quanta, four and one-half hours.—By Professor G. RICCI: Absolute differential calculus, potential, elasticity, four hours.—By Professor F. SEVERI: Differential geometry, four hours.—By Professor A. SIGNORINI: Elasticity with technical applications, three hours.—

By Professor G. VERONESE: Foundations of geometry, four hours.

UNIVERSITY OF PALERMO.—By Professor G. BAGNERA: Theory of automorphic functions, modular functions, three hours.—By Professor M. GEBBIA: Electricity and magnetism, four and one-half hours.—By Professor G. B. GUCCIA: General theory of algebraic curves and surfaces, four and one-half hours.—By Professor A. VENTURI: Potential, form of celestial bodies, elementary theory of the tides, four and one-half hours.

UNIVERSITY OF PAVIA.—By Professor L. BERZOLARI: Birational transformations in the plane and in the space with applications, three hours.—By Professor U. CISOTTI: Potential, propagation of heat, three hours.—By Professor F. GERBALDI: Functions of a complex variable, elliptic functions, three hours.—By Professor G. VIVANTI: Theory of transformation groups, three hours.

UNIVERSITY OF PISA.—By Professor E. BERTINI: Geometry on an algebraic curve, three hours.—By Professor L. BIANCHI: Curves, surfaces, and curved three-dimensional spaces, four and one-half hours.—By Professor U. DINI: Complements of infinitesimal analysis, integral equations, four and one-half hours.—By Professor G. A. MAGGI: Abstract of potential theory, formation and properties of the equations of elastic motion with application to theoretical optics, formation and properties of the equations of the electromagnetic field, electromagnetic theory of light, four and one-half hours.—By Professor P. PIZZETTI: Interpolation formulae, principles of spherical astronomy, mechanical theory of the shape of celestial bodies, four and one-half hours.

UNIVERSITY OF ROME.—By Professor L. AMOROSO: Theory of function of a complex variable and of elliptic functions, three hours.—By Professor G. BISCONCINI: Geometrical applications of infinitesimal analysis, three hours.—By Professor G. CASTELNUOVO: Topics having a bearing on elementary mathematics, abelian functions, three hours.—By Professor L. SILLA: Elasticity with technical applications, three hours.—By Professor V. VOLTERRA: Thermodynamics,

three hours.—Mechanical problems as application of the functions of lines and of their analysis, three hours.

UNIVERSITY OF TURIN.—By Professor T. BOGGIO: Analytic dynamics, three hours.—By Professor G. FUBINI: Ordinary differential equations, classic and recent results, three hours.—By Professor G. SANNIA: Ruled surfaces, investigation of congruences and complexes of rays by means of differential quadratic forms determining them, two hours.—By Professor C. SEGRE: Selected topics of hypergeometry, three hours.—By Professor C. SOMIGLIANA: Electricity and optics, three hours.

THE following courses are announced in mathematics for the winter semester 1913–1914:

UNIVERSITY OF BERLIN.—By Professor H. A. SCHWARZ: Analytic geometry, four hours; Synthetic geometry, four hours; Applications of elliptic functions, one hour; Colloquium, four hours; Seminar, two hours.—By Professor G. FROBENIUS: Theory of numbers, four hours; Seminar, two hours.—By Professor F. SCHOTTKY: Theory of elliptic functions, four hours; Theory of space curves and surfaces, four hours.—By Professor R. LEHMANN-FILHÉS: Integral calculus, four hours; Determinants, four hours.—By Professor G. HETTNER: Theory of probabilities, two hours.—By Professor J. KNOBLAUCH: Differential calculus, four hours; Elliptic functions, four hours; Quadrature of the circle, one hour.—By Dr. K. KNOPP: Theory of functions, II, four hours; Algebra, four hours.—By Professor O. KRIGAR-MENZEL: Mechanics of a particle, four hours.

UNIVERSITY OF BONN.—By Professor E. STUDY: Differential geometry, two hours; Higher geometry, one hour; Theory of functions, four hours; Seminar, two hours.—By Professor F. LONDON: Analytic geometry, four hours; Synthetic geometry, with exercises, three hours; Seminar, two hours.—By Professor I. SCHUR: Differential and integral calculus, four hours; Linear substitutions and elementary divisors, two hours.—By Dr. J. O. MÜLLER: Theory of determinants, two hours; Calculus of variations, three hours.

UNIVERSITY OF GÖTTINGEN.—By Professor D. HILBERT: Analytic mechanics, four hours; Electromagnetic oscillations, two hours.—By Professor W. VOIGT: Theory of the potential, four hours.—By Professor C. RUNGE: Differential and integral calculus, six hours; Mechanics of continua, two hours.—By Professor L. PRANDTL; Hydrodynamics, three hours.—By Professor E. LANDAU: Theory of numbers, four hours; Seminar, two hours.—By Professor C. CARATHÉODORY: Projective geometry, four hours; Conformal depiction, four hours; Seminar, two hours.—By Professor F. BERNSTEIN: Mathematical statistics and insurance, three hours; Theory of probabilities, three hours.—By Professor O. TOEPLITZ: Partial differential equations, four hours; Elementary mathematics, four hours.—By Dr. R. COURANT: Infinite series with applications, four hours.—By Dr. M. BORN: Elements of mathematical physics, four hours.—By Dr. E. HECKE: Historical development of the fundamental concepts of mathematics, two hours; Exercises in mechanics, two hours.—By Dr. L. v. SENDEN: Graphical statics, three hours; Vector analysis, two hours.

UNIVERSITY OF LEIPZIG.—By Professor K. ROHN: Analytic geometry of space, with exercises, five hours; Determinants, two hours; Seminar, two hours.—By Professor O. HÖLDER: Differential and integral calculus, with exercises, six hours; Partial differential equations, two hours; Seminar, two hours.—By Professor G. HERGLOTZ: Differential geometry, three hours; Mechanics, five hours; Seminar, two hours.—By Professor A. v. OETTINGEN: Geometric-perspective drawing, one hour.—By Professor P. KOEBE: Elliptic functions with applications, five hours; Seminar, two hours.—By Dr. R. KONIG; Higher algebra, two hours.

UNIVERSITY OF MUNICH.—By Professor F. LINDEMANN: Differential and integral calculus, five hours; Differential geometry, four hours; Geometry of the line and sphere, two hours; Seminar, two hours.—By Professor A. PRINGSHEIM: Foundations of arithmetic and analysis, four hours; Elliptic functions, four hours; Seminar, two hours.—By Professor A. VOSS; Algebra, four hours; Introduction to the theory of ordinary differential equations, four hours; Seminar, two hours.—By Professor F. HARTOGS: Descriptive geometry,

with exercises, seven hours; Trigonometry with applications, two hours.—By Professor H. BRUNN: Elements of higher mathematics, four hours.—By Dr. H. DINGLER: Elementary mathematics from a higher point of view, four hours; History of mathematics, two hours; Foundations of mathematics, two hours.—By Dr. F. BÖHM: Plane analytic geometry, four hours; Problems of life insurance, four hours; Life insurance companies, one hour.—By Dr. A. ROSENTHAL: Synthetic geometry, four hours; Proseminar, two hours.

UNIVERSITY OF STRASSBURG.—By ———: Differential and integral calculus, four hours.—By Professor F. SCHUR: Analytic geometry of the plane and of space, four hours; Selected chapters of differential geometry, two hours; Seminar, two hours.—By Professor M. SIMON: History of mathematics in antiquity, two hours.—By Professor J. WELLSTEIN: Introduction to the theory of functions and of elliptic functions, four hours.—By Professor R. v. MISES: Mechanics, I, four hours; Graphical and numerical methods of integration, two hours; Seminar, two hours.—By Professor S. EPSTEIN: Determinants and invariants, two hours.—By Dr. A. SPEISER: Fourier series, two hours.

PROFESSOR G. CANTOR, of the University of Halle, and Professor G. RODENBERG, of the technical school at Hannover, have been decorated with the order of the crown of the third class.

PROFESSOR M. DEHN, of the University of Kiel, has been appointed professor of mathematics at the technical school at Breslau.

DR. R. WEITZENBÖCK and DR. P. ROTH have been appointed docents in mathematics at the University of Vienna.

DR. E. STEINITZ has been appointed docent in mathematics at the University of Breslau.

DR. C. THAER has been appointed docent in mathematics at the University of Greifswald.

DR. L. BIEBERBACH, of the University of Königsberg, has

accepted a professorship of mathematics at the University of Bern.

DR. S. DUMAS, of the Swiss bureau of insurance, has been appointed associate professor of the mathematics of finance at the University of Lausanne.

DR. W. MATHIES, of the University of Münster, has been appointed professor of mathematical physics at the University of Basel.

DR. H. WEYL, of the University of Göttingen, has been appointed professor of higher mathematics at the technical school of Zurich, as successor to Professor C. F. GEISER, who has retired. Dr. G. DUMAS has been promoted to an associate professorship of mathematics at Zurich.

PROFESSOR M. ABRAHAM, of the technical school at Milan, has been promoted to a full professorship of rational mechanics

PROFESSOR G. SCORZA, of the University of Cagliari, has been transferred to the professorship of projective and descriptive geometry at the University of Parma.

DR. L. AMOROSO has been appointed docent in mathematics at the University of Rome.

DR. G. BORDIGA, of the University of Padua, has been promoted to an associate professorship of projective geometry.

DR. E. DANIELE, of the University of Pavia, has been appointed associate professor of mathematical physics at the University of Catania.

DR. L. TONELLI, of the University of Bologna, has been appointed associate professor of analysis at the University of Cagliari.

MR. S. B. MCLAREN, lecturer of mathematics at the University College, Reading, has been appointed professor of mathematics at the same institution.

At the University of Chicago Professors G. A. Bliss and H. E. Slaught have been promoted to full professorships of mathematics.

Dr. N. J. Lennes, of Columbia University, has been appointed professor and head of the department of mathematics in the University of Montana.

At Stanford University, Professor H. F. Blichfeldt has been promoted to a full professorship of mathematics and Professor W. A. Manning has been promoted to an associate professorship of applied mathematics.

At the University of Oregon Dr. W. M. Smith has been promoted to a full professorship of mathematics. Dr. R. M. Winger, of the University of Illinois, has been appointed to an assistant professorship of mathematics.

At Northwestern University, Dr. E. J. Moulton has been appointed assistant professor of mathematics. Dr. C. R. Dines and Mr. P. M. Bachelder have been appointed instructors in mathematics.

Dr. L. L. Dines, of Columbia University, has been appointed associate professor and acting head of the department of mathematics in the University of Arizona.

Professor F. Anderegg, of Oberlin College, has been granted a year's leave of absence to study in Europe.

Dr. Henry Blumberg has been appointed instructor in mathematics in the University of Nebraska.

Professor W. P. Webber, of Bethany College, has been appointed instructor in mathematics in the University of Pittsburgh.

Dr. T. H. Brown has been appointed instructor in mathematics at the Sheffield Scientific School of Yale University.

At the University of Minnesota Dr. W. E. Anderson, Dr. W. O. Beal, and Dr. W. L. Miser have been appointed instructors in mathematics.

DR. MILDRED L. SANDERSON has been appointed instructor in mathematics at the University of Wisconsin.

DR. G. W. HESS has accepted the professorship of mathematics at Shurtleff College, Alton, Illinois.

DR. D. M. SMITH and Dr. D. L. STAMPY have been appointed instructors in mathematics at the Georgia School of Technology.

DR. R. E. ROOT, of the University of Missouri, has been appointed instructor in mathematics in the U. S. Naval Academy.

AT Dartmouth College Professor R. M. BARTON, who has been absent on leave for two years, has resigned. Mr. M. G. GABA has gone to the University of Chicago for further study. Mr. E. S. ALLEN has been appointed instructor in mathematics.

PROFESSOR TH. FRIESSENDORFF, of the technical school at St. Petersburg, died March 6, at the age of 42 years.

PROFESSOR J. KÖNIG, of the University of Budapest, died April 8, at the age of 63 years.

PROFESSOR J. G. WHITE, of Kentucky University, died July 18, at the age of 67 years.

PROFESSOR C. G. ROCKWOOD, of Princeton University, died on July 2, at the age of seventy-one years. He had been a member of the American Mathematical Society since 1891.

LUCIAN AUGUSTUS WAIT, emeritus professor of mathematics at Cornell University, died September 6, 1913, at the age of 67 years. He had been a member of the American Mathematical Society since 1891, and was a member of the Council for six years.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ACKERMANN (R.). Böschungsstrahlen und Böschungsflächen. (Diss.) Halle, 1913.

ADRIAN (P.). Untersuchungen über eine Funktion, die eine partikuläre Lösung einer linearen homogenen Differentialgleichung 2ter Ordnung ist. (Diss.) Bern, 1912. 8vo. 65 pp.

BERLINER (H.). Theorie der Polaren in Bezug auf Dreiecke in synthetischer Behandlung. Bern, 1912. 8vo. 94 pp.

BRUGNATELLI (L.). Matematica allegra, con appendice, non matematica, ma abbastanza allegra. Milano, Saita, 1912. 8vo. 92 pp. L. 2.00

COPPEDÈ (C.). Classificazione topologica delle superficie di Lamé algebriche. Firenze, Calasanziana, 1912. 8vo. 46 pp.

CRUDELI (U.). Compendio ed esercizî di calcolo differenziale e di calcolo integrale. Milano, Albrighi, Segati e C., 1913. 8vo. 8 + 243 pp. L. 4 50

CULLIS (C. E.). Matrices and determinoids. Volume I. (University of Calcutta Readership Lectures.) Cambridge, University Press, 1913. 8vo. 21 s.

FABRY (E.). Problèmes et exercices de mathématiques générales. 2e édition revue et très augmentée. Paris, Hermann, 1913. 8vo. 496 pp. Fr. 10.00

GUGGISBERG (K.). Die Konchoide der kubischen Parabel. (Diss.) Bern, 1912. 8vo. 39 pp.

HAS (A. R.). Das Fermatsche Prinzip. Darmstadt, 1912. 8vo. 18 pp. M. 2.00

HATTON (J. L. S.). The principles of projective geometry applied to the straight line and conic. Cambridge, University Press, 1913. 8vo. 376 pp. 10 s. 6d.

HEINIS (H.). Ueber eine Gleichung 5ten Grades der komplexen Multiplikation der elliptischen Modulfunktionen. (Diss.) Basel, 1911. 8vo. 78 pp.

JOURDAIN (P. E. B.). The nature of mathematics. (The people's books, No. 94.) London, Jack, 1913.

KAEGBEIN (P.). Die Gleichung $x^m + y^m = z^m$ und der Fermatsche Satz. Hamburg, 1913.

KNESER (A.). Mathematik und Natur. (Rektorats-Rede.) 2ter Abdruck. Breslau, Trewendt und Granier, 1913. 8vo. 18 pp. M. 0.50

KRAFFT (G.). Ueber die merkwürdigen Punkte des Tetraeders im nichteuklidischen Raume. (Diss.) Strassburg, 1912. 8vo. 37 pp.

LIETZMANN (W.) und TRIER (V.). Wo steckt der Fehler? Trugschlüsse und Schülerfehler. (Mathematische Bibliothek. Nr. 10.) Leipzig, Teubner, 1913. 8vo. 4 + 57 pp. M. 0.80

LILIENTHAL (R. v.). Vorlesungen über Differentialgeometrie. 2ter Band: Flächentheorie. 1ter Teil. (Teubner's Sammlung. 28ter Band, II.) Leipzig, Teubner, 1913. 8vo. 8 + 270 pp. Cloth. M. 13.00

MICHELSON (P.). Die bestimmten algebraischen Gleichungen des 1–4 Grades. Nebst einem Anhang: Unbestimmte Gleichungen. 3te Auflage. Hannover, Meyer, 1913. 8vo. 8 + 306 pp. Cloth. M. 4.50

NICODEMI (R.). Piani quotati. Napoli, Pironti, 1913. 8vo. 73 pp. L. 5.50

——. Spostamento di una figura piana nel suo piano. Napoli, Pironti, 1913. 8vo. 59 pp. L. 2.00

PEZZO (P. Del). Principi di geometria proiettiva; lezioni dettate nell'università di Napoli. 3a edizione. Napoli, De Rubertis, 1913. 8vo. 431 pp. L. 14.00

POLETTI (L.). La tavola dei numeri primi per l'intervallo da 10,000,000 à 10,020,000. Pontremoli, Cavanna, 1913. 8vo. 8 pp.

RUSSELL (B.). See WHITEHEAD (A. N.).

SCHENKER (O.). Das Dreieck und die Kiepert'sche Parabel. (Diss.) Bern, 1912. 8vo. 52 pp.

SCHLUSSER (H.). Die Fokaltheorie der linearen Strahlenkongruenzen. (Diss.) Strassburg, 1912. 8vo. 49 pp.

SCHMID (A.). Abzählungen bezüglich der Ebene im n-dimensionalen Raum in algebraischer Behandlung. (Diss.) Tübingen, 1912. 8vo. 26 pp.

VOGT (H.). Eléments de mathématiques supérieures. Edition réduite, à l'usage des élèves chimistes. Paris, Vuibert et Nony, 1913. 8vo. 6 + 357 pp.

VOSS (A.). Ueber das Wesen der Mathematik. (Rede.) Erweitert und mit Anmerkungen versehen. 2te, sorgfältig durchgesehene und vermehrte Auflage. Leipzig, Teubner, 1913. 8vo. 4 + 123 pp. M. 4.00

WHITEHEAD (A. N.) and RUSSELL (B.). Principia mathematica. Volume III: Theory of series (continued); theory of measurement. Cambridge, University Press, 1913. 8vo. 21 s.

WILSON (H.). Untersuchung einer linear-quadratischen Berührungstransformation. (Diss., Rostock.) Weida i. Thür., Thomas & Hubert, 1913.

ZÜHLKE (P.). Konstruktionen in begrenzter Ebene. (Mathematische Bibliothek, Nr. 11.) Leipzig, Teubner, 1913. 8vo. 4 + 40 pp. M. 0.80

II. ELEMENTARY MATHEMATICS.

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